UNIVERSITY OF KRAGUJEVAC
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CONTRIBUTIONS TO THE THEORY OF SHIFT-INVARIANT SPACES

Doctoral Dissertation

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PRILOZI TEORIJI TRANSLACIONO-INVARIJANTNIH PROSTORA

doktorska disertacija

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Preface

This doctoral dissertation reports on the results obtained during a few years of work under the advisement of Academician Stevan Pilipović and Prof. Suzana Aleksić (papers [6, 7, 8]). The research area of this dissertation is shift-invariant spaces of Sobolev type. This research area belongs to functional analysis (harmonic analysis and microlocal analysis).

During these years, I have had the great honor and pleasure to work with Prof. Pilipović. The experience, I gained from working with him is an invaluable asset. I am grateful for his great support and help he gave me. His advice and views related both to mathematics and writing scientific papers, as well as to life, were priceless for me. He was always available despite his numerous commitments. On this occasion, I would also like to thank Prof. Suzana Aleksić for my "mathematical" awakening, great support and her helpful advice. Namely, back in 2011, when she started working as a professor at the Faculty of Science, Prof. Suzana singled out the students who excelled at the preliminary examination in the scope of the course Analysis 2. She worked with them additionally and discussed mathematical problems and assertions. I was one of those four students. Until that moment I had only dealt with tasks and had studied theory only as much as it was enough, but that extra work inspired me to take mathematics more seriously. I would also like to thank my aunts, sisters, brothers, and relatives for their understanding and support.

Also, my great gratitude goes to Prof. Nenad Teofanov, the leader of the project GOALS - *Global and local analysis of operators and distributions*, #2727, for giving me the opportunity to work in this very promising team. I feel thank him for many useful and important conversations. Finally, I would like to express my special thanks to the Science Found of the Republic of Serbia for their great contribution to the accomplishment of this doctoral dissertation.

I dedicate this dissertation to my father Slobodan (1946–2001) and mother Koviljka (1954–2015).

Science Fund
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Kragujevac, August 2024 — Aleksandar Aksentijević

Abstract

This doctoral dissertation investigates shift-invariant subspaces V_r of Sobolev spaces $H^r(\mathbb{R}^n)$, where $r \in \mathbb{R}$. Characterization of the spaces V_r was performed using range functions, range operators, shift-preserving operators, and wave front. Also, characterizations of frames, Riesz families, and Bessel families were performed using the mentioned operators and especially using Gram's and dual Gram's matrix. Relationships between the mentioned operators were investigated, and the conditions under which the shift-preserving operator could be s-diagonalizable and could be written as a finite sum of products of its s-eigenvalues and corresponding projections were determined. The problem of dynamical sampling for spaces V_r was solved and different approaches to the theory of shift-invariant spaces were identified. Elements of the spaces V_r were described using a wave front. Finally, conditions under which there exists a product of elements from the observed spaces and conditions when such a product would belong to some shift-invariant space were determined.

The dissertation consists of six chapters. The first chapter is of an introductory nature. It consists of a brief overview of the achieved results in the space $L^2(\mathbb{R}^n)$ including the focus on the importance of shift-invariant spaces and other concepts mentioned in dissertation. The second chapter presents the theory of distributions. The main tool used in dissertation, the Fourier transform, is presented in the third chapter. Also, Sobolev spaces $H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, and spaces $\mathscr{D}_{L^2}(\mathbb{R}^n)$, $\mathscr{D}'_{L^2}(\mathbb{R}^n)$, are presented in the third chapter. The fourth chapter discusses spaces of periodic functions and periodic distributions, some important equalities used in research, and the theory of wave fronts. Theory of frames in Hilbert spaces is presented in the fifth chapter. Finally, the sixth chapter presents original results of this dissertation.

Key words: Sobolev spaces, shift-invariant space, range function, range operator, shift-preserving operator, frame, s-diagonalization, dynamical sampling, wave front, product of distributions.

Apstrakt

Ova doktorska disertacija istražuje translaciono-invarijantne potprostore V_r prostora Soboljeva $H^r(\mathbb{R}^n)$, pri čemu je $r \in \mathbb{R}$. Karakterizacija prostora V_r izvršena je korišćenjem funkcije opsega, operatora opsega, operatora koji komutiraju sa translacijama i talasnim frontom. Takođe, izvršena je karakterizacija okvira, Risove familije i Beselove familije uz pomoć pomenutih operatora i posebno koristeći Gramovu i dualnu Gramovu matricu. Istraživani su odnosi između navedenih operatora i određeni uslovi pod kojima operator koji komutira sa translacijama može biti s-dijagonalizabilan i može se zapisati kao konačan zbir proizvoda njegovih s-sopstvenih vrednosti i odgovarjućih projekcija. Problem dinamičkog uzorkovanja za prostore V_r je rešen i povezani su različiti pristupi teoriji translaciono-invarijantnih prostora. Elementi prostora V_r su opisani pomoću talasnog fronta. Na kraju, uslovi pod kojima postoji proizvod elemenata iz posmatranih prostora i uslovi kada će takav proizvod pripadati nekom translaciono-invarijantnom prostoru su određeni.

Disertaciju čini šest glava. Prva glava je uvodnog karaktera. Sastoji se iz kratkog pregleda postignutih rezultata u prostoru $L^2(\mathbb{R}^n)$, uključujući i fokus na značaj translaciono-invarijantnih prostora i drugih pojmova koji se pominju u disertaciji. U drugoj glavi izložena je teorija distribucija. Glavni alat koji se koristi u disertaciji, Furijeova transformacija, predstavljena je u trećoj glavi. Takođe, prostori Soboljeva $H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, i prostori $\mathscr{D}_{L^2}(\mathbb{R}^n)$, $\mathscr{D}'_{L^2}(\mathbb{R}^n)$ su predstavljeni u trećoj glavi. Četvrta glava sadrži prostore periodičnih funkcija i periodičnih distribucija, neke bitne jednakosti koje se koriste u istraživanju, i teoriju o talasnom frontu. Teorija okvira u Hilbertovim prostorima je izložena u petoj glavi. Na kraju, u šestoj glavi su predstavljeni originalni rezultati ove disertacije.

Ključne reči: Soboljevi prostori, translaciono-invarijantni prostori, funkcija opsega, operator opsega, operator koji komutira sa translacijama, okvir, s-dijagonalizacija, dinamičko uzorkovanje, talasni front, proizvod distribucija.

Contents

Chapter 1

Introduction

The role of mathematical research in solving a large number of problems in education and society is growing amazingly. Harmonic analysis constitutes a leading part in resolving these problems. Harmonic analysis is a branch of mathematics that arose from ancient attempts to display functions as superpositions of some elementary functions with oscillatory ones, i.e. wave nature. The term ”harmonics” comes from the ancient Greek word meaning ”skilled in music”. In the physical problems of eigenvalues, this term has come to denote waves whose frequencies are integer multiples of a fundamental frequency, such as the harmonic frequencies of musical notes. However, this term has been generalized over time beyond its original meaning.

A modern branch of harmonic analysis is time-frequency analysis. Time-frequency analysis includes parts of mathematics and applied mathematics that use time-frequency shifts (translations and modulations) for analysis of operators. It is a form of local Fourier analysis that simultaneously and symmetrically treats time and frequency. Time-frequency analysis has a wide range of applications: in physics, signal analysis, engineering, image processing, communication theory, quantum mechanics, etc.

This dissertation specifically studies the Sobolev[1] spaces of functions that are invariant under translation, i.e. shift-invariant spaces of Sobolev type. The advantage of shift-invariant spaces is reflected in the fact that the simplicity and structure of the space are maintained, so they are more flexible for approximating real data. They are used in the finite element method, approximation theory, the construction of multiresolution approximations, spaces of signals and images, wavelet theory, etc. (see [10, 13, 14, 18, 40, 42, 44, 57, 73]).

The structure of shift-invariant subspaces of space $L^2 = L^2(\mathbb{R}^n)$ was first studied by Marcin Bownik, in 1999. In the paper [23], using the range function, the range operator and shift-preserving operator, he provides a characterization of frames such that checking whether $E(\mathscr{A}_I) = \{T_q f : f \in \mathscr{A}_I, q \in \mathbb{Z}^n, \mathscr{A}_I \subset L^2\} \subset L^2$ is a frame on a ”large” subspaces of L^2 reduces to the problem of checking it on a ”small” subspaces of $\ell^2(\mathbb{Z}^n)$, where I is finite set or $I = \mathbb{N}$, and $T_q f(\cdot) = f(\cdot - q)$. In this way, the problem of determining whether a set of functions is a frame or a Riesz[2] family in large subspaces

[1]Sergei Lvovich Sobolev (1908–1989) – Russian mathematician.
[2]Frigyes Riesz (1880–1956) – Hungarian mathematician.

of L^2 is reduced by switching to the Fourier[3] domain and a small subspace of the space ℓ^2 parameterized by $\mathbb{T}^n = [-\frac{1}{2}, \frac{1}{2})^n$. Therefore, the analysis of frames and Riesz families using the Gram's[4] matrix and its dual matrix is simplified. It is proved that every (even infinitely dimensional) shift-invariant space can be decomposed into an orthogonal sum of spaces, each generated by a single function whose shifts form a Parseval[5] frame for that space. By applying this fact, the characterization of shift-preserving operators in the sense of range operators is given, and some facts about the dimension function are proved. Some important results of M. Bownik are presented in more detail below since they stimulated this research.

Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean[6] space. This is space of all n-tuples $x = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$. Similarly, the notations $\mathbb{Z}^n$ and $\mathbb{N}_0^n$ will be used for the corresponding n-tuples. The inner product on $\mathbb{R}^n$ is $\langle x, y\rangle = \sum_{k=1}^n x_k y_k$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$. Also, throughout this dissertation, the n-dimensional integral will be denoted by $\int_\Omega f(x)\,\mathrm{d}x$, where $\Omega \subseteq \mathbb{R}^n$. Recall that L^s-norm of a measurable function f is

$$\|f\|_{L^s} = \left(\int_{\mathbb{R}^n} |f(x)|^s \,\mathrm{d}x\right)^{1/s},$$

where $s \in [1, +\infty)$. If $\|f\|_{L^s} < +\infty$, then $f \in L^s(\mathbb{R}^n)$. The space $L^s(\mathbb{R}^n)$ is a Banach space. A measurable function f belongs to $L^\infty(\mathbb{R}^n)$ if

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| < +\infty,$$

i.e. if f is essentially bounded. The space $L^2(\mathbb{R}^n)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,\mathrm{d}x, \quad f, g \in L^2(\mathbb{R}^n).$$

The short notations L^s and L^∞ will be used for spaces $L^s(\mathbb{R}^n)$, $s \in [1, +\infty)$, and $L^\infty(\mathbb{R}^n)$, respectively. Further, let $\mathscr{A}$ be at most countable set of functions from L^2 and $E(\mathscr{A}) = \{T_q f : f \in \mathscr{A},\, q \in \mathbb{Z}^n\}$. In the following, I denotes a finite set or $I = \mathbb{N}$ (unless otherwise stated). Therefore, the notation $\mathscr{A}_I$ will also be used when an index set I is given. If $I = \{1, \ldots, m\}$, then the notation $\mathscr{A}_m$ will be used. An arbitrary Hilbert[7] space will be denoted by $\mathscr{H}$. By $\|\cdot\|$ will be denoted the operator norm. The imaginary unit is denoted by i ($i^2 = -1$).

Definition 1.0.1 ([43])**.** *A (closed) subspace $V \subset \mathscr{H}$ for which holds*

$$f \in V \text{ implies } T_q f \in V \text{ for every } q \in \mathbb{Z}^n,$$

is said to be a shift-invariant (SI) space.

Bownik in [23] observed $\mathscr{H} = L^2$ and used $S(\mathscr{A}) = \overline{\operatorname{span}}\, E(\mathscr{A})$ to denote the SI space generated by $\mathscr{A} \subset L^2$, where $\overline{\operatorname{span}}\, E(\mathscr{A})$ denotes the closed set of all linear combinations of vectors $E(\mathscr{A})$. Next, he introduces a new space.

[3]Jean-Baptiste Joseph Fourier (1768–1830) – French mathematician and physicist.

[4]Jørgen Pedersen Gram (1850–1916) – Danish actuary and mathematician.

[5]Marc-Antoine Parseval des Chênes (1755–1836) – French mathematician.

[6]Euclid (325 BCE–265 BCE) – an ancient Greek mathematician active as a geometer and logician; "the father of geometry".

[7]David Hilbert (1862–1943) – German mathematician.

Definition 1.0.2 ([23]). *The space of all vector valued measurable functions $G : \mathbb{T}^n \to \ell^2$ such that*

$$\int_{\mathbb{T}^n} \|G(t)\|_{\ell^2}^2 \,\mathrm{d}t < +\infty$$

is denoted by $L^2(\mathbb{T}^n, \ell^2)$.

Lemma 1.0.1 ([23]). *The space $L^2(\mathbb{T}^n, \ell^2)$ is a Hilbert space with the inner product*

$$\langle G_1, G_2 \rangle_{L^2(\mathbb{T}^n,\ell^2)} = \int_{\mathbb{T}^n} \langle G_1(t), G_2(t) \rangle_{\ell^2}^2 \,\mathrm{d}t,$$

and the corresponding norm

$$\|G\|_{L^2(\mathbb{T}^n,\ell^2)} = \left(\int_{\mathbb{T}^n} \|G(t)\|_{\ell^2}^2 \,\mathrm{d}t \right)^{1/2}.$$

Further, he uses the Fourier transform defined by

$$\mathscr{F} f(t) = \mathscr{F}[f](t) = \widehat{f}(t) = \int_{\mathbb{R}^n} f(x) \,\mathrm{e}^{-2\pi i \langle x,t \rangle} \,\mathrm{d}x, \quad t \in \mathbb{R}^n,$$

and introduces two important mappings.

Lemma 1.0.2 ([23]). *The mapping $\mathscr{T} : L^2 \to L^2(\mathbb{T}^n, \ell^2)$ defined by*

$$\mathscr{T} f(t) = \left(\widehat{f}(t+q)\right)_{q \in \mathbb{Z}^n}, \quad t \in \mathbb{T}^n, \ f \in L^2,$$

is an isometric isomorphism. Moreover, for every $f \in L^2$ holds $\mathscr{T} f(\cdot - q) = \mathrm{e}^{-2\pi i \langle q, \cdot \rangle} \mathscr{T} f(\cdot)$, $q \in \mathbb{Z}^n$.

Definition 1.0.3 ([23]). *A mapping*

$$J : \mathbb{T}^n \to \left\{ \text{closed subspaces of } \ell^2 \right\}$$

($t \mapsto J(t)$, $t \in \mathbb{T}^n$) is called the range function.

The range function J is measurable if for any $a, b \in \ell^2$, $t \mapsto \langle P_{J(t)}(a), b \rangle_{\ell^2}$ is a measurable scalar function (i.e. if $P_{J(t)}$, $t \in \mathbb{T}^n$, are weakly operator measurable), where $P_{J(t)} : \ell^2 \to J(t)$, $t \in \mathbb{T}^n$, are the associated orthogonal projections. Range functions are said to be equal if they are equal almost everywhere (or a.e. for short).

In the following theorem he connects SI spaces with the range function and vice versa. Bownik got the idea for this claim from Helson's[8] book [43].

Theorem 1.0.1 ([23]). *A space $V \subset L^2$ is SI if and only if there is a measurable range function J so that*

$$V = \left\{ f \in L^2 \,:\, \mathscr{T} f(t) \in J(t) \text{ for a.e. } t \in \mathbb{T}^n \right\}.$$

The relationship between V and J is one-to-one. If $V = S(\mathscr{A}_I)$, then

$$J(t) = \overline{\operatorname{span}} \{ \mathscr{T} f(t) : f \in \mathscr{A}_I \}.$$

[8]Henry Berge Helson (1927–2010) – American mathematician.

Some authors call $\mathscr{T}f(t)$, i.e. $\big(\widehat{f}(t+q)\big)_{q\in\mathbb{Z}^n}$ the fiber for function f at t, and the space $J(t)$ the fiber space for V at t (see [4, 5]).

The idea for the characterization of frames is implicitly observed in the work [68]. For the notation in the theorems 1.0.2 and 1.0.3, the reader can refer to Chapter 5.

Theorem 1.0.2 ([23]). *Let $V = S(\mathscr{A}_I)$. Then, $E(\mathscr{A}_I)$ is*

(1) *a frame of V with frame bounds A and B if and only if $\{\mathscr{T}f(t) : f \in \mathscr{A}_I\} \subset \ell^2$ is a frame of $J(t)$ with frame bounds A and B for a.e. $t \in \mathbb{T}^n$;*

(2) *a Riesz family (basis) of V with bounds A and B if and only if $\{\mathscr{T}f(t) : f \in \mathscr{A}_I\} \subset \ell^2$ is a Riesz family (basis) of $J(t)$ with bounds A and B for a.e. $t \in \mathbb{T}^n$;*

(3) *a Bessel family of V with bound B if and only if $\{\mathscr{T}f(t) : f \in \mathscr{A}_I\} \subset \ell^2$ is a Bessel family of $J(t)$ with bound B for a.e. $t \in \mathbb{T}^n$;*

(4) *a fundamental frame of V if and only if $\{\mathscr{T}f(t) : f \in \mathscr{A}_I\} \subset \ell^2$ is a fundamental frame of $J(t)$ for a.e. $t \in \mathbb{T}^n$.*

Bownik then introduces the definition of dimension function, the definition of spectrum of SI space and proves the decomposition theorem.

Definition 1.0.4 ([23]). *Let $V = S(\mathscr{A}_I)$ and J be the corresponding range function.*

(1) *A mapping $\dim_V : \mathbb{T}^n \to \mathbb{N} \cup \{0, +\infty\}$ defined by $\dim_V(t) = \dim J(t)$ is called the dimension function of V.*

(2) *The spectrum of space V is defined by $\sigma_V = \big\{t \in \mathbb{T}^n : \dim J(t) > 0\big\}$ or equivalently $\sigma_V = \big\{t \in \mathbb{T}^n : J(t) \neq \{\mathbf{0}\}\big\}$.*

Theorem 1.0.3 (**The decomposition theorem,** [23]). *Let V be a SI subspace of L^2. Then, V can be decomposed into an orthogonal sum, i.e.*

$$V = \bigoplus_{k\in\mathbb{N}} S(f_k),$$

such that $\{T_q f_k : q \in \mathbb{Z}^n\}$ is a tight frame of $S(f_k)$ and $\sigma_{S(f_{k+1})} \subset \sigma_{S(f_k)}$ for every $k \in \mathbb{N}$. Moreover, $\dim_{S(f_k)}(t) = \|\mathscr{T}f_k(t)\|_{\ell^2}$ for every $k \in \mathbb{N}$, and

$$\dim_V(t) = \sum_{k\in\mathbb{N}} \|\mathscr{T}f_k(t)\|_{\ell^2} \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Furthermore, he introduces another very important operator, the range operator. This operator is defined as a family of operators, and gives a very significant connection between range operators and shift-preserving operators. An operator L is shift-preserving if it is linear and bounded and commutes with translations.

Definition 1.0.5 ([23]). *An operator defined on J (with values in ℓ^2) by*

$$R : \mathbb{T}^n \to \{\text{bounded operators defined on closed subspaces of } \ell^2\},$$

such that the domain of $R(t)$ is $J(t)$ for a.e. $t \in \mathbb{T}^n$, is called the range operator. The range operator R is measurable if $t \mapsto R(t)P_{J(t)}$, $t \in \mathbb{T}^n$, is a weakly measurable operator.

Theorem 1.0.4 ([23]). *Assume that $V \subset L^2$ is a SI space and J is its associated range function.*

(1) *If $L : V \to L^2$ is a shift-preserving operator, then there is a measurable range operator R on J so that*

$$(\mathscr{T}L)f(t) = R(t)(\mathscr{T}f(t)) \quad \text{for a.e. } t \in \mathbb{T}^n,\ f \in V. \tag{1.0.1}$$

(2) *If R is a measurable range operator on J so that $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|R(t)\| < +\infty$, then there is a shift-preserving operator $L : V \to L^2$ so that (1.0.1) holds.*

The correspondence between L and R is one-to-one and $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|R(t)\| = \|L\|$.

Further, Bownik uses these obtained results and gives the properties of the dimension function and determines a dual frame for a given frame. M. Bownik's work was followed by A. Aguilera and collaborators. They continued to study the range function, shift-preserving operators, and range operators [4, 5]. They introduced the definition of s-diagonalization and determined conditions under which the shift-preserving operator L could be s-diagonalizable and represented by using a finite sum of products s-eigenvalues of the operator L and the corresponding orthogonal projections. Also, Aguilera and collaborators dealt with the problem of dynamical sampling for shift-preserving operators defined on SI subspaces of L^2.

Dynamic sampling deals with the problem of reconstructing a signal from its samples. That is, it is necessary to determine the conditions for a bounded operator $T : \mathscr{H} \to \mathscr{H}$ defined on a Hilbert space $\mathscr{H}$ and a set of functions $\mathcal{F} = \{f_j : j \in J\} \subset \mathscr{H}$ so that the set $\{T^k f_j : j \in J,\ k \in E\}$ is a basis or frame for $\mathscr{H}$, whereby the index sets J and E are the subsets of $\mathbb{N}_0$. In this way, it is possible to compensate the lack of information for the signal f, by sampling the signals Tf, $T^2 f$, $T^3 f$, ... This problem has recently attracted a lot of attention from mathematicians and there are different interpretations of it [1, 2, 3, 9, 11, 12, 20, 50].

Furthermore, the mentioned results and papers are followed by papers [21, 24, 25, 55, 60] and many others. In the dissertation, all the important results of Bownik and Aguilera, papers [23] and [4, 5], are extended to SI spaces of the Sobolev type (the sections 6.1–6.5 and 6.7–6.9). Further, using the Fourier transform, an additional structure of SI spaces was obtained (Section 6.6). On the other hand, there was also a somewhat different approach to SI spaces, such as the approach of the authors in [15] and [64]. They consider SI spaces of form

$$\mathscr{V}_r = \left\{ f : f = \sum_{k=1}^{m} \sum_{q\in\mathbb{Z}^n} \alpha_{q,k} T_q f_k,\ (\alpha_{q,k})_{q\in\mathbb{Z}^n} \in \ell_r^2,\ f_k \in \mathscr{L}^\infty \cap L_r^2,\ k = 1, \dots, m \right\}. \tag{1.0.2}$$

Recall that the space of weighted sequence ℓ_r^s is defined by

$$\ell_r^s = \ell_r^s(\mathbb{Z}^n) = \left\{ (\alpha_q)_{q\in\mathbb{Z}^n} : \sum_{q\in\mathbb{Z}^n} |\alpha_q|^2 \mu_r^s(q) < +\infty \right\}, \quad s \geqslant 1,\ r \in \mathbb{R},$$

where $\mu_r(\cdot) = (1 + |\cdot|^2)^{r/2}$. Obviously, $\mu_r^s = \mu_{sr}$ for $s \in \mathbb{R}$. Note, the space ℓ_r^2 is a Hilbert space with the inner product

$$\big\langle (\alpha_{q,1})_{q\in\mathbb{Z}^n}, (\alpha_{q,2})_{q\in\mathbb{Z}^n} \big\rangle_{\ell_r^2} = \sum_{q\in\mathbb{Z}^n} \alpha_{q,1} \overline{\alpha_{q,2}} \mu_{2r}(q).$$

The space $\mathscr{L}^\infty$ defined by

$$\mathscr{L}^\infty = \left\{ f : \|f\|_{\mathscr{L}^\infty} = \sup_{t \in \mathbb{T}^n} \sum_{q \in \mathbb{Z}^n} |T_q f(t)| < +\infty \right\}$$

is a subspace of L^2 (see [15, 64]), and the space L^2_r is defined by

$$L^2_r = L^2_r(\mathbb{R}^n) = \left\{ f : \int_{\mathbb{R}^n} |f(t)|^2 \mu_{2r}(t) \, \mathrm{d}t < +\infty \right\}, \quad r \in \mathbb{R}. \tag{1.0.3}$$

The space

$$\mathscr{V}_0 = \left\{ f : f = \sum_{k=1}^{m} \sum_{q \in \mathbb{Z}^n} \alpha_{q,k} T_q f_k, \ (\alpha_{q,k})_{q \in \mathbb{Z}^n} \in \ell^2, \ f_k \in \mathscr{L}^\infty, \ k = 1, \dots, m \right\}$$

is analyzed in [15].

The following two statements are the main results of papers [15] and [64].

Theorem 1.0.5 ([15])**.** *Let $\mathscr{M}_m = \{f_k : f_k \in \mathscr{L}^\infty, \, k = 1, \dots, m\}$. Then, the following statements are equivalent.*

(1) *$\mathscr{V}_0$ is closed in L^2.*

(2) *$E(\mathscr{M}_m)$ is a frame for $\mathscr{V}_0$.*

Theorem 1.0.6 ([64])**.** *Let $\mathscr{K}_m = \{f_k : f_k \in \mathscr{L}^\infty \cap L^2_r, \, k = 1, \dots, m\}$. Then, the following statements are equivalent.*

(1) *$\mathscr{V}_r$ is closed in L^2_r.*

(2) *$E(\mathscr{K}_m)$ is a frame for $\mathscr{V}_r$.*

An important difference is that with this approach to SI spaces, the sequence of coefficients belongs to space ℓ^2_r. In the dissertation, these two approaches to SI spaces are connected (Section 6.6). Moreover, Pilipović and Simić in paper [64] observe spaces $\mathscr{V}^s_r$ with sequences from ℓ^s_r, but in this dissertation only the case $s = 2$ is significant.

In the further research of SI spaces of Sobolev type, this dissertation uses the wave front of Sobolev type introduced by Hörmander[9] in [46], and results of paper [56] in which Maksimović, Pilipović and collaborators performed the discretization of the wave front of Sobolev type of a distribution $f \in \mathscr{D}'(\mathbb{R}^n)$ in terms of Fourier series coefficients.

A wave front (or a wave front set) is a term that appeared in the period of research related to the classification of singularities by means of their spectrum and it is at the base of microlocal analysis (microlocal analysis is a part of analysis in which properties of distributions are studied). Until the late 1990s, wave front rarely appeared when solving physics problems. During the 1990s, the wave front set was proved to be a crucial in defining quantum fields in curved space-times, Dirac[10] fields, quantization of gravity, etc., followed by the intense studies of different types of wave front sets. Hörmander's concept of the wave front (set) [46]–[48] has attracted the mathematicians' attention and there

[9]Lars Valter Hörmander (1931–2012) – Swedish mathematician.

[10]Paul Adrien Maurice Dirac (1902–1984) – English theoretical physicist.

is extensive literature on it and its important role in the qualitative analysis of partial differential equations and pseudo-differential operators.

Using the discretization of the wave front of Sobolev type from [56], the elements of the observed spaces will be described in the dissertation. Also, conditions are obtained under which the product of two elements from different SI spaces exists, and moreover belongs to some SI space (the sections 6.10 and 6.11).

Before presenting all the obtained results, the chapters 2–5 will cover the necessary theoretical framework for a better understanding of the noted results. Therefore, let us take a short trip through the theoretical background of the topic.

Chapter 2

Theory of distributions

The theory of distributions (theory of general functions) was created with desire to find a correct mathematical approach to the mathematical models of various processes, which were not clearly based (mathematically speaking), and enable mathematical solutions that will have a natural sense.

The concept of functions and operations with functions in the classical analysis, due to its pronounced narrowness, did not always enable an adequate solution of those models. This resulted in several attempts to generalize the notion of function and operations with it. The results of Sobolev [70] and [71] have the most prominent place. In the monograph "Théorie des distributions" (1950/1951) L. Schwartz[1] was the first who publish a systematized theory of one class of general functions – distributions (the latest edition [69] was issued in 1966).

The theory of distributions represents a mathematical tool for various areas of mathematical physics, the theory of partial differential equations, harmonic analysis, the theory of pseudo-differential and Fourier operators. Its applications can be found in [19, 26, 32, 34, 37, 75] and many other papers and books.

In classical analysis, continuous functions do not have to be differential. Distributions are, roughly speaking, a generalization of the concept of functions so that every continuous function is a distribution. Its derivative is not a function, it is a distribution. Moreover, every distribution is differentiable and its derivative is a distribution. For example, in physics, one comes across quantities that have a very large value in a very small domain, but are equal to zero outside of it. In 1926, Dirac introduced a mathematical notation for such cases by defining the δ-distribution, which is also called the Dirac delta distribution. It is defined by

$$\delta_0(x) = \begin{cases} +\infty, & x = 0, \\ 0, & x \neq 0, \end{cases} \quad \text{and} \quad \int_{\mathbb{R}} \delta_0(x)\,\mathrm{d}x = 1.$$

This led to birth of a new theory, the theory of distributions.

In this chapter, the basic definitions and properties that are needed for further work will be listed, for details you can see [16, 36, 37, 38, 45, 49, 65, 69, 74, 75].

[1] Laurent-Moïse Schwartz (1915–2002) – French mathematician.

2.1 The spaces $\mathscr{D}(\mathbb{R}^n)$ and $\mathscr{D}'(\mathbb{R}^n)$

Let us introduce the basic notation that will be used in this chapter and in the dissertation. If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}_0^n$, then

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

and $|x|^s = |x_1|^s + |x_2|^s + \dots + |x_n|^s$, $s \in [1, +\infty)$. For every $a = (a_1, a_2, \dots, a_n) \in \mathbb{N}_0^n$,

$$\partial^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}} = D^a.$$

The set $\Omega \subseteq \mathbb{R}^n$ will denote the open set, and $\mathscr{C}(\Omega)$ the set of continuous functions on Ω. The label $\overline{\Omega}$ indicates the closure of the set Ω.

Definition 2.1.1 ([65])**.** *The set* $\operatorname{supp}\phi = \overline{\{x \in \Omega : \phi(x) \neq 0\}}$ *is called the support of the function* $\phi \in \mathscr{C}(\Omega)$.

Using relation $\overline{A \cup B} = \overline{A} \cup \overline{B}$, for sets $A, B \subseteq \mathbb{R}^n$, the next assertion follows.

Lemma 2.1.1 ([45, 65])**.** *Let* $\phi, \psi \in \mathscr{C}(\mathbb{R}^n)$. *Then,* $\operatorname{supp}(\phi + \psi) \subseteq \operatorname{supp}\phi \cup \operatorname{supp}\psi$, *and* $\operatorname{supp}(C\phi) = \operatorname{supp}\phi$ *for every* $C \in \mathbb{C}\backslash\{0\}$.

Definition 2.1.2 ([65])**.** *Let* $\ell \in \mathbb{N}_0$ *or* $\ell = +\infty$. *The set* $\mathscr{C}^\ell(\Omega)$ *denotes the set of functions that are defined over* Ω *and have all continuous derivatives up to order* ℓ. *The set* $\mathscr{C}_0^\ell(\Omega)$ *is a subset of* $\mathscr{C}^\ell(\Omega)$ *of those functions whose supports are compact in* Ω.

Note, if $\ell = 0$, then $\mathscr{C}^0(\Omega) = \mathscr{C}(\Omega)$.

Remark 2.1.1. (1) *Since every compact set in* Ω *is also compact in* $\mathbb{R}^n$, *it follows that* $\mathscr{C}_0^\ell(\Omega) \subseteq \mathscr{C}_0^\ell(\mathbb{R}^n)$.

(2) *The spaces* $\mathscr{C}^\ell(\Omega)$ *and* $\mathscr{C}_0^\ell(\Omega)$ *are vector spaces over the field of complex numbers.*

(3) *The functions from* $\mathscr{C}^\infty(\Omega)$ *are called smooth functions.*

Example 2.1.1. *The function* $\phi(x) = \begin{cases} \mathrm{e}^{(|x|^2-1)^{-1}}, & |x| < 1 \\ 0, & |x| \geqslant 1 \end{cases}$, *given on Figure* 1 *belongs to the space* $\mathscr{C}_0^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp}\phi = \mathcal{K}[0,1]$, *where* $\mathcal{K}[0,1]$ *is the closed ball with center at zero and radius* 1.

A set A is said to be a convex set if $\alpha A + \beta A \subseteq A$ holds for any $\alpha \geqslant 0$ and $\beta \geqslant 0$ such that $\alpha + \beta = 1$.

Definition 2.1.3 ([45, 58, 65])**.** (1) *A vector space* W *over the scalar field* $\mathbb{K} = \{\mathbb{C}, \mathbb{R}\}$ *provided with a topology is called a topological vector space if the mappings* $(x, y) \mapsto x + y \in W$ *and* $(\lambda, x) \mapsto \lambda x \in W$ *are continuous, where* $x, y \in W$, $\lambda \in \mathbb{K}$.

(2) *A topological vector space that has a neighborhood base at* $\mathbf{0}$ *composed of convex sets is called a locally convex space.*

Recall, a topological vector space W is a locally convex space if and only if the topology of the space W is given by a family of seminorms.

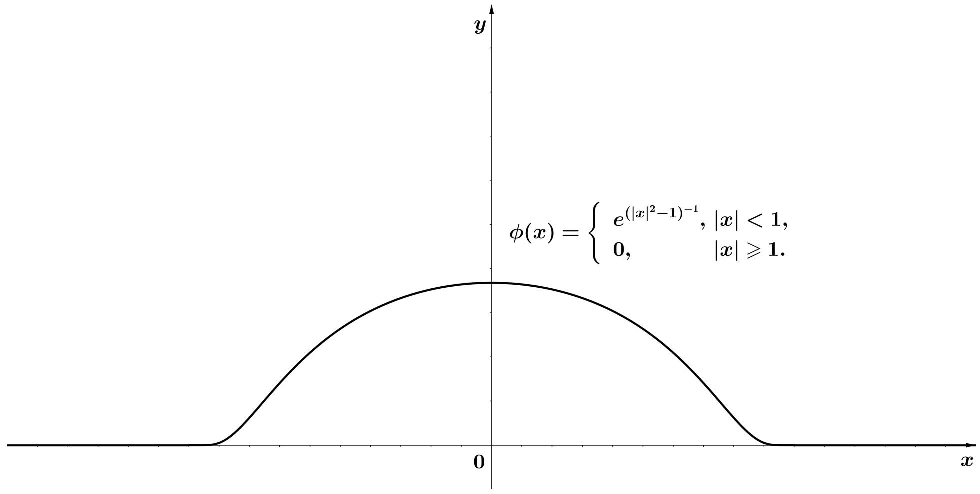

Figure 1.

Let $K \subset \Omega$ be a compact set and $\ell \in \mathbb{N}_0$ or $\ell = +\infty$. The space $\mathscr{C}_0^\ell(K)$ is a subspace of $\mathscr{C}_0^\ell(\Omega)$ whose elements have supports contained in K. Moreover, the space $\mathscr{C}_0^\infty(K)$ with the family of norms

$$p_{K,\ell}(\phi) = \sum_{|a| \leqslant \ell} \sup_{x \in K} |D^a \phi(x)|, \quad \ell \in \mathbb{N}_0,$$

is a locally convex space. The sets

$$U_{K,\ell,m} = \left\{ \phi \in \mathscr{C}_0^\infty(K) : p_{K,\ell}(\phi) < \frac{1}{m} \right\}, \quad m \in \mathbb{N},\, \ell \in \mathbb{N}_0,$$

form a neighborhood base at $\mathbf{0}$.

Definition 2.1.4 ([65]). *The vector space $\mathscr{C}_0^\infty(K)$ equipped with the given topology is a locally convex space $\mathscr{D}(K)$.*

If K and $\widetilde{K}$ are compact sets such that $K \subset \widetilde{K}$, then $\mathscr{D}(K) \subset \mathscr{D}(\widetilde{K})$ and the topology in $\mathscr{D}(K)$ coincides with the topology which $\mathscr{D}(\widetilde{K})$ induces on $\mathscr{D}(K)$. For every non-empty open set $\Omega \subset \mathbb{R}^n$ an increasing sequence of compact sets $K_j \subset \Omega$, $j \in \mathbb{N}$, such that (for every $j \in \mathbb{N}$)

$$K_j \subset \operatorname{Int} K_{j+1} \quad \text{and} \quad \Omega = \bigcup_{j \in \mathbb{N}} K_j$$

can be constructed, where $\operatorname{Int} K_j$ is the interior of K_j. Thus, $\mathscr{D}(\Omega) = \bigcup_{j \in \mathbb{N}} \mathscr{D}(K_j)$. Let the space $\mathscr{D}(\Omega)$ be equipped with the finest topology for which all the canonical injections $i_j : \mathscr{D}(K_j) \to \mathscr{D}(\Omega)$, $j \in \mathbb{N}$, are continuous.

Definition 2.1.5 ([45]). *The space $\mathscr{D}(\Omega)$ is called the space of test functions.*

Remark 2.1.2. *The space $\mathscr{C}_0^\infty(\Omega)$ is often denoted by $\mathscr{D}(\Omega)$.*

Some properties of the space of test functions are stated in the following theorems.

Theorem 2.1.1 ([65]). *A linear mapping T of space $\mathscr{D}(\Omega)$ into a locally convex space is continuous if and only if T is continuous over $\mathscr{D}(K)$ for every compact set $K \subset \Omega$.*

Theorem 2.1.2 ([65]). *For sequence $(\psi_\nu)_{\nu \in \mathbb{N}} \in \mathscr{D}(\Omega)$ holds $\psi_\nu \to \psi$ in $\mathscr{D}(\Omega)$ if and only if there is a compact set $K \subset \Omega$ such that for every $\nu \in \mathbb{N}$, $\operatorname{supp} \psi_\nu \subset K$, and there is a test function ψ such that for every $a \in \mathbb{N}_0^n$, $D^a \psi_\nu$ converges uniformly to $D^a \psi$.*

Theorem 2.1.3 ([45]). *The space $\mathscr{D}(\Omega)$ is a complete space.*

Proof. Fix $j \in \mathbb{N}$. Since the space $\mathscr{D}(K_j)$ is complete, it follows that it is closed in $\mathscr{D}(K_{j+1})$. Thus, $\mathscr{D}(\Omega)$ is a complete space. □

Definition 2.1.6 ([65]). *A continuous linear functional on the space of test functions is called a distribution. The set of all distributions defined on Ω is denoted by $\mathscr{D}'(\Omega)$.*

The action of a distribution f on test function ψ is denoted by (f, ψ), i.e. $f : \psi \to (f, \psi)$. Thus, $\mathscr{D}'(\Omega)$ is the dual space of $\mathscr{D}(\Omega)$.

Convergence in $\mathscr{D}'(\Omega)$ is given by the next definition.

Definition 2.1.7 ([75]). *A sequence $(f_\nu)_{\nu\in\mathbb{N}}$ from $\mathscr{D}'(\Omega)$ converges to $f \in \mathscr{D}'(\Omega)$, i.e. $\lim_{\nu\to+\infty} f_\nu = f$ in $\mathscr{D}'(\Omega)$, if $\lim_{\nu\to+\infty}(f_\nu, \psi) = (f, \psi)$ for every $\psi \in \mathscr{D}(\Omega)$.*

Some important properties of distributions are given in the following theorems.

Theorem 2.1.4 ([45, 65]). (1) *The space $\mathscr{D}'(\Omega)$ is a complete space.*

(2) *The space $\mathscr{D}(\Omega)$ is dense in the space $\mathscr{D}'(\Omega)$.*

Theorem 2.1.5 ([65]). *A linear functional f defined on $\mathscr{D}(\Omega)$ belongs to $\mathscr{D}'(\Omega)$ if and only if for every sequence $(\psi_\nu)_{\nu\in\mathbb{N}} \in \mathscr{D}(\Omega)$ such that $\psi_\nu \to \mathbf{0}$ in $\mathscr{D}(\Omega)$ it follows that $(f, \psi_\nu) \to 0$ in $\mathbb{C}$, as $\nu \to +\infty$.*

Theorem 2.1.6 ([65]). *A linear functional f on the space $\mathscr{D}(\Omega)$ is a distribution (i.e. belongs to $\mathscr{D}'(\Omega)$) if and only if for every compact set $K \subset \Omega$ there is a constant $C > 0$ and $\ell \in \mathbb{N}_0$ such that for all $\psi \in \mathscr{D}(\Omega)$ with support contained in K holds*

$$|(f, \psi)| \leqslant C p_{K,\ell}(\psi). \tag{2.1.1}$$

Proof. Let $(\psi_\nu)_{\nu\in\mathbb{N}} \in \mathscr{D}(\Omega)$ such that $\psi_\nu \to \mathbf{0}$ in $\mathscr{D}(\Omega)$ as $\nu \to +\infty$. Then, $(f, \psi_\nu) \to 0$ in $\mathbb{C}$, by (2.1.1). Thus, according to Theorem 2.1.5, $f \in \mathscr{D}'(\Omega)$.

For the opposite implication, suppose that $f \in \mathscr{D}'(\Omega)$ and that (2.1.1) is not valid. Then, there is a sequence $(\psi_\nu)_{\nu\in\mathbb{N}} \in \mathscr{D}(K)$ for some compact set K, such that

$$|(f, \psi_\nu)| > \nu p_{K,\nu}(\psi_\nu), \quad \nu \in \mathbb{N}. \tag{2.1.2}$$

Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be given by $\phi_\nu = \frac{\psi_\nu}{(f,\psi_\nu)}$, $\nu \in \mathbb{N}$. Then, $p_{K,\nu}(\phi_\nu) = \frac{p_{K,\nu}(\psi_\nu)}{|(f,\psi_\nu)|} \leqslant \frac{1}{\nu}$, $\nu \in \mathbb{N}$, by (2.1.2). Obviously, for every $\ell \leqslant \nu$ holds $p_{K,\ell}(\phi_\nu) \leqslant p_{K,\nu}(\phi_\nu) \leqslant \frac{1}{\nu}$. Thus, $\phi_\nu \to \mathbf{0}$ in $\mathscr{D}(K)$. But, on the other hand $(f, \phi_\nu) = 1$ for every $\nu \in \mathbb{N}$, i.e. $(f, \phi_\nu) \nrightarrow 0$ in $\mathbb{C}$. This contradicts the fact that $f \in \mathscr{D}'(\Omega)$. Therefore, (2.1.1) is valid. □

Remark 2.1.3. *Theorem* 2.1.6 *is used in some papers and books (monographs) to define the distribution (for example see* [46]*).*

Example 2.1.2. *Let $x_0 \in \Omega$. The Dirac distribution $\delta_{x_0}(\cdot) = \delta_0(\cdot - x_0)$ is given by*

$$\big(\delta_{x_0}, \psi\big) = \psi(x_0), \quad \psi \in \mathscr{D}(\Omega).$$

Indeed, since

$$\begin{aligned} C_1\big(\delta_{x_0}, \psi_1\big) + C_2\big(\delta_{x_0}, \psi_2\big) &= C_1\psi_1(x_0) + C_2\psi_2(x_0) \\ &= \big(\delta_{x_0}, C_1\psi_1 + C_2\psi_2\big), \quad C_1, C_2 \in \mathbb{C},\ \psi_1, \psi_2 \in \mathscr{D}(\Omega), \end{aligned}$$

it follows that δ_{x_0} is a linear functional. Further, let $K \subset \Omega$ be a compact set. Then,

$$|(\delta_{x_0}, \psi)| = |\psi(x_0)| \leqslant p_{K,\ell}(\psi) \quad \text{for every } \psi \in \mathscr{D}(K).$$

Thus, by Theorem 2.1.6, δ_{x_0} is a distribution.

Definition 2.1.8 ([46, 65])**.** *The singular support of $f \in \mathscr{D}'(\Omega)$, denoted by $\operatorname{sign\,supp} f$, is the set of points in Ω that do not have a neighborhood where f is a smooth function.*

In other words, the singular support is the complement of the union of all open sets in Ω over which f is a smooth function. Thus, $\operatorname{sign\,supp} f$ is closed in Ω.

Theorem 2.1.7 ([65])**.** *The distribution f is a smooth function in the complement of* $\operatorname{sign\,supp} f$.

The next theorem gives the conditions under which distribution $f \in \mathscr{D}'(\Omega)$ can be extended to $f_0 \in \mathscr{D}'(\mathbb{R}^n)$.

Theorem 2.1.8 ([65])**.** *A distribution $f \in \mathscr{D}'(\Omega)$ can be extended to $\mathscr{D}'(\mathbb{R}^n)$ if and only if for every point $\widetilde{x} \in \overline{\Omega} \setminus \Omega$ there are an open neighborhood $U_{\widetilde{x}}$ of point $\widetilde{x}$, a constant C and $\ell \in \mathbb{N}_0$ so that*

$$|(f, \psi)| \leqslant C \sum_{|a| \leqslant \ell} \sup_{x \in \Omega \cap U_{\widetilde{x}}} |D^a \psi(x)|, \quad \psi \in \mathscr{C}_0^\infty(\Omega \cap U_{\widetilde{x}}).$$

In the continuation, the shorter terms $\mathscr{D} = \mathscr{D}(\mathbb{R}^n)$, $\mathscr{D}' = \mathscr{D}'(\mathbb{R}^n)$, $\mathscr{C} = \mathscr{C}(\mathbb{R}^n)$, $\mathscr{C}_0 = \mathscr{C}_0(\mathbb{R}^n)$ and $\mathscr{C}_0^\infty = \mathscr{C}_0^\infty(\mathbb{R}^n)$ will be used, where $\mathscr{C}_0 = \mathscr{C}_0(\mathbb{R}^n)$ is the Banach[2] space of continuous functions vanishing at infinity.

The motivation for the definition of derivative of distribution will be noted in Section 2.3.

Definition 2.1.9 ([75])**.** *The derivative $D^a f$, $a \in \mathbb{N}_0^n$, of $f \in \mathscr{D}'$ is defined by*

$$(D^a f, \psi) = (-1)^{|a|}(f, D^a \psi), \quad \psi \in \mathscr{D}.$$

Theorem 2.1.9 ([75])**.** *The mapping $D^a : \mathscr{D}' \to \mathscr{D}'$, $a \in \mathbb{N}_0^n$, is continuous.*

Proof. Let $\lim_{\nu \to +\infty} f_\nu = f$ in $\mathscr{D}'$. Then,

$$\lim_{\nu \to +\infty} (D^a f_\nu, \psi) = \lim_{\nu \to +\infty} (-1)^{|a|}(f_\nu, D^a \psi) = (-1)^{|a|}(f, D^a \psi) = (D^a f, \psi), \quad \psi \in \mathscr{D},$$

by the definitions 2.1.7 and 2.1.9. Hence, $\lim_{\nu \to +\infty} D^a f_\nu = D^a f$ in $\mathscr{D}'$. □

[2]Stefan Banach (1892–1945) – Polish mathematician, one of the most influential mathematicians of the 20th century; the founder of modern functional analysis.

2.2 The spaces $\mathscr{E}(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$

In this dissertation, spaces $\mathscr{E}(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$ will only be defined, but readers interested further on this can refer to [45, 53, 65, 75].

Let a family of seminorms $(q_{K,a})_{K,a}$ be defined by

$$q_{K,a}(\psi) = \max_{x \in K} |D^a \psi(x)|, \quad \psi \in \mathscr{C}^\infty(\Omega), \tag{2.2.1}$$

where K passes through compact subsets of Ω and $a \in \mathbb{N}_0^n$.

Definition 2.2.1 ([45])**.** *The subspace of the space $\mathscr{C}^\infty(\Omega)$ equipped with a locally convex topology induced by the family of seminorms* (2.2.1) *is denoted by $\mathscr{E}(\Omega)$.*

Remark 2.2.1. (1) *The notation $\mathscr{E}(\Omega)$ is also used for $\mathscr{C}^\infty(\Omega)$.*

(2) *The space $\mathscr{D}(\Omega)$ contains those $\psi \in \mathscr{E}(\Omega)$ whose* $\operatorname{supp}\psi$ *is compact.*

Definition 2.2.2 ([65])**.** *The space of continuous linear functionals defined over $\mathscr{E}(\Omega)$ is denoted by $\mathscr{E}'(\Omega)$.*

Theorem 2.2.1 ([53, 65])**.** (1) *The space $\mathscr{D}$ is dense in $\mathscr{E} = \mathscr{E}(\mathbb{R}^n)$.*

(2) *The space $\mathscr{E}' = \mathscr{E}'(\mathbb{R}^n)$ is a subspace of $\mathscr{D}'$.*

Theorem 2.2.2 ([45])**.** *Let $f \in \mathscr{D}'(\Omega)$. Then, $f \in \mathscr{E}'(\Omega)$ if and only if the set* $\operatorname{supp} f \subset \Omega$ *is compact.*

The space $\mathscr{E}'(\Omega)$ can be identified with the subspace of distributions $\mathscr{E}'$ whose elements have a compact support contained in Ω.

2.3 The spaces $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$

The space $\mathscr{S}(\mathbb{R}^n)$ is better known as the Schwartz space. This space is very useful in Fourier analysis. It contains smooth functions $\psi(x)$, $x \in \mathbb{R}^n$, which together with all its derivatives decay rapidly when $|x| \to +\infty$.

Definition 2.3.1 ([65])**.** *The vector space $\mathscr{S}(\mathbb{R}^n)$ ($\mathscr{S}$ for short) is the set of functions $\psi \in \mathscr{C}^\infty$ such that*

$$q_{a,b}(\psi) = \sup_{x \in \mathbb{R}^n} \left| x^a D^b \psi(x) \right| < +\infty \tag{2.3.1}$$

for any $a, b \in \mathbb{N}_0^n$.

Example 2.3.1. *The function $\psi(x) = \mathrm{e}^{-|x|^2} \in \mathscr{S}$, but $\psi \notin \mathscr{C}_0^\infty$.*

From the previous example, it follows that the Schwartz space is larger than the space $\mathscr{C}_0^\infty$. Moreover,

$$\mathscr{C}_0^\infty \subset \mathscr{S} \subset \mathscr{C}^\infty.$$

The space $\mathscr{S}$ has the following properties.

Theorem 2.3.1 ([49, 53, 67, 75, 77])**.** (1) *The embedding $\mathscr{D} \hookrightarrow \mathscr{S}$ is continuous.*

(2) *The space $\mathscr{D}$ is dense in $\mathscr{S}$.*

(3) *The space* $\mathscr{S}$ *is dense in* $\mathscr{E}$.

(4) *The space* $\mathscr{S}$ *is complete.*

(5) *If* $\psi_1, \psi_2 \in \mathscr{S}$, *then* $\psi_1\psi_2 \in \mathscr{S}$. *Moreover, the space* $\mathscr{S}$ *is: closed under linear combinations; closed under multiplication by polynomials; closed under differentiation; closed under translations and multiplication by* $\mathrm{e}^{i\langle x,t\rangle}$.

Instead of the family of seminorms (2.3.1), it is sometimes convenient to use the family of seminorms

$$q'_{a,b}(\psi) = \int_{\mathbb{R}^n} |x^a D^b \psi(x)| \, \mathrm{d}x \tag{2.3.2}$$

or

$$q''_{a,b}(\psi) = \left(\int_{\mathbb{R}^n} |x^a D^b \psi(x)|^2 \, \mathrm{d}x \right)^{1/2}. \tag{2.3.3}$$

Lemma 2.3.1 ([53, 65])**.** *The families of seminorms* (2.3.1), (2.3.2) *and* (2.3.3) *are equivalent. Also, the families of seminorms*

$$\widetilde{q}_{c,b}(\psi) = \sup_{x \in \mathbb{R}^n} \left| (1+|x|^2)^{c/2} D^b \psi(x) \right|$$

and

$$q_c(\psi) = \sup_{x \in \mathbb{R}^n, |b| \leqslant c} \left| (1+|x|^2)^{c/2} D^b \psi(x) \right|$$

are equivalent with (2.3.1), *where* $c \in \mathbb{N}_0$.

Note, if the families of seminorms are equivalent, then they define the same topology.

In the space $\mathscr{S}$, the convergence is defined as follows.

Definition 2.3.2 ([75])**.** *It is said that a sequence* $(\psi_\nu)_{\nu \in \mathbb{N}}$ *from* $\mathscr{S}$ *converges to* $\psi \in \mathscr{S}$, *i.e.* $\lim_{\nu \to +\infty} \psi_\nu = \psi$ *in* $\mathscr{S}$, *if*

$$\lim_{\nu \to +\infty} q_{a,b}(\psi_\nu - \psi) = 0 \quad \textit{for any } a, b \in \mathbb{N}_0^n.$$

Therefore, it is obvious that if a sequence converges in the space of test functions $\mathscr{D}$, then it also converges in Schwartz space.

Definition 2.3.3 ([51])**.** *A function* f *is of slow qrowth if there are constants* $C > 0$, $s \geqslant 0$, *and* $A > 0$ *so that*

$$|D^a f(x)| \leqslant C|x|^s, \quad |x| > A,$$

for every $a \in \mathbb{N}_0^n$.

Definition 2.3.4 ([65, 75])**.** *The space of continuous linear functionals on* $\mathscr{S}$ *is the space of tempered distributions (generalized functions of slow growth). It is denoted by* $\mathscr{S}'(\mathbb{R}^n)$ *or simply* $\mathscr{S}'$.

Hence, $\varphi \in \mathscr{S}'$ if and only if $\varphi : \mathscr{S} \to \mathbb{C}$ is linear and $\lim_{\nu \to +\infty} \psi_\nu = \psi$ in $\mathscr{S}$ implies that $\lim_{\nu \to +\infty} (\varphi, \psi_\nu) = (\varphi, \psi)$ in $\mathbb{C}$.

Theorem 2.3.2 (**L. Schwartz,** [75])**.** *Let* φ *be a linear functional over* $\mathscr{S}$. *Then,* $\varphi \in \mathscr{S}'$ *if and only if there are* $C > 0$ *and* $c \in \mathbb{N}_0$ *such that*

$$|(\varphi, \psi)| \leqslant C q_c(\psi) \tag{2.3.4}$$

for every $\psi \in \mathscr{S}$.

Proof. Let $\varphi \in \mathscr{S}'$ and assume that (2.3.4) is not valid, i.e. C and c do not exist. Then, there is a sequence of functions $(\psi_\nu)_{\nu\in\mathbb{N}}$ which belong to $\mathscr{S}$ so that

$$|(\varphi, \psi_\nu)| \geqslant \nu q_\nu(\psi_\nu), \quad \nu \in \mathbb{N}. \tag{2.3.5}$$

Define the sequence $(\phi_\nu)_{\nu\in\mathbb{N}}$ to be

$$\phi_\nu(x) = \frac{\psi_\nu(x)}{\nu^{1/2} q_\nu(\psi_\nu)}, \quad \nu \in \mathbb{N}.$$

Then, $\lim_{\nu\to+\infty} \phi_\nu = \mathbf{0}$ in $\mathscr{S}$, and for $\nu \geqslant \max\{|a|, |b|\}$,

$$|x^a D^b \phi_\nu(x)| = \frac{|x^a D^b \psi_\nu(x)|}{\nu^{1/2} q_\nu(\psi_\nu)} \leqslant \frac{C_1}{\nu^{1/2}},$$

by Lemma 2.3.1. From this, since $\varphi \in \mathscr{S}'$, it follows that

$$\lim_{\nu\to+\infty} (\varphi, \phi_\nu) = 0. \tag{2.3.6}$$

But, from (2.3.5),

$$|(\varphi, \phi_\nu)| = \frac{|(\varphi, \psi_\nu)|}{\nu^{1/2} q_\nu(\psi_\nu)} \geqslant \sqrt{\nu},$$

contrary to (2.3.6).

Now, let φ be the linear functional over $\mathscr{S}$ which satisfies (2.3.4) for some $C > 0$ and $c \in \mathbb{N}_0$. Let $(\psi_\nu)_{\nu\in\mathbb{N}}$ be a sequence such that $\lim_{\nu\to+\infty} \psi_\nu = \psi$ in $\mathscr{S}$. Then,

$$\lim_{\nu\to+\infty} q_c(\psi_\nu - \psi) = 0$$

and thus $\lim_{\nu\to+\infty}(\varphi, \psi_\nu) = (\varphi, \psi)$, by (2.3.4). Hence, $\varphi \in \mathscr{S}'$. □

In the following definition, the convergence in $\mathscr{S}'$ is given.

Definition 2.3.5 ([75]). *It is said that a sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ from $\mathscr{S}'$ converges to $\varphi \in \mathscr{S}'$, i.e. $\lim_{\nu\to+\infty} \varphi_\nu = \varphi$ in $\mathscr{S}'$, if $\lim_{\nu\to+\infty}(\varphi_\nu, \psi) = (\varphi, \psi)$ for every $\psi \in \mathscr{S}$.*

From the definitions, it follows that $\mathscr{S}' \subset \mathscr{D}'$ and if the sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ converges in $\mathscr{S}'$, it implies that $(\varphi_\nu)_{\nu\in\mathbb{N}}$ also converges in $\mathscr{D}'$. Moreover, the space $\mathscr{S}'$ has the following properties.

Theorem 2.3.3 ([45, 49, 53, 65]). (1) *The spaces $\mathscr{D}$ and $\mathscr{S}$ are dense in $\mathscr{S}'$.*

(2) *If $\varphi \in \mathscr{S}'$, then its restriction on $\mathscr{D}$ belongs to the space $\mathscr{D}'$, i.e. $\varphi_{\restriction\mathscr{D}} \in \mathscr{D}'$.*

(3) *Hold*

$$\mathscr{D} \subset \mathscr{S} \subset \mathscr{E} \quad \text{and} \quad \mathscr{E}' \subset \mathscr{S}' \subset \mathscr{D}', \tag{2.3.7}$$

with continuous imbeddings.

Theorem 2.3.4 ([49, 53]). *The spaces $\mathscr{D}$ and $\mathscr{S}$ are dense in L^s, $s \in [1, +\infty)$. Moreover, the space L^s, $s \in [1, +\infty]$, is dense in $\mathscr{S}'$.*

Theorem 2.3.5 ([51]). *If a function f is of slow growth, then it generates a distribution by the formula*

$$(f, \psi) = \int_{\mathbb{R}^n} f(x)\psi(x)\,\mathrm{d}x, \quad \psi \in \mathscr{S}.$$

Proof. It is not difficult to see that it is a linear functional. Let $(\psi_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $\mathscr{S}$ such that $\lim_{\nu\to+\infty} \psi_\nu = \mathbf{0}$ in $\mathscr{S}$. Then, for sufficiently large $c \geqslant 0$,

$$
\begin{aligned}
|(f,\psi_\nu)| &= \left|\int_{\mathbb{R}^n} f(x)\psi_\nu(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^n} \frac{f(x)}{(1+|x|^2)^c}\cdot(1+|x|^2)^c\psi_\nu(x)\,\mathrm{d}x\right| \\
&\leqslant \sup\left|(1+|x|^2)^c\psi_\nu(x)\right| \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|^2)^c}\,\mathrm{d}x \to 0, \quad \text{as} \quad \nu\to+\infty.
\end{aligned}
$$

Thus, $\lim_{\nu\to+\infty}(f,\psi_\nu)=0$, i.e. the linear functional is continuous. □

The spaces $\mathscr{D}'$, $\mathscr{E}'$ and $\mathscr{S}'$ are weakly complete in the following sense.

Theorem 2.3.6 ([53, 72])**.** *Let $\mathscr{X}$ be $\mathscr{D}$, $\mathscr{E}$ or $\mathscr{S}$, and let $\mathscr{X}'$ be its dual space. The space $\mathscr{X}'$ is (weakly) complete, i.e. if $(f_\nu)_{\nu\in\mathbb{N}}$ is a sequence from $\mathscr{X}'$ such that $((f_\nu,\psi))_{\nu\in\mathbb{N}}$ is a Cauchy*[3] *sequence for every $\psi\in\mathscr{X}$, then there exists $\lim_{\nu\to+\infty} f_\nu = f$ in $\mathscr{X}'$.*

Now, the motivation for introducing the definition of derivative of distributions in the space $\mathscr{D}'$ (Definition 2.1.9) follows. Let $f\in\mathscr{D}'(\Omega)$ and $\psi\in\mathscr{D}(\Omega)$. Since the support of the function ψ is contained in some compact set $K\subset\Omega$, using partial integration, it gives

$$
(D^a f,\psi) = \int_{\mathbb{R}^n} D^a f(x)\psi(x)\,\mathrm{d}x = (-1)^{|a|}\int_{\mathbb{R}^n} f(x)D^a\psi(x)\,\mathrm{d}x = (-1)^{|a|}(f,D^a\psi).
$$

Also, this equality is taken to define the derivative of tempered distributions.

Definition 2.3.6 ([75])**.** *The derivative $D^a\varphi$, $a\in\mathbb{N}_0^n$, of $\varphi\in\mathscr{S}'$ is defined by*

$$
(D^a\varphi,\psi) = (-1)^{|a|}(\varphi,D^a\psi), \quad \psi\in\mathscr{S}.
$$

Theorem 2.3.7 ([75])**.** *If $\varphi\in\mathscr{S}'$, then $D^a\varphi\in\mathscr{S}'$, $a\in\mathbb{N}_0^n$.*

Proof. Since $D^a:\mathscr{S}\to\mathscr{S}$ is continuous, the right-hand side of the equation in Definition 2.3.6 is a linear continuous functional over $\mathscr{S}$. Thus, $D^a\varphi\in\mathscr{S}'$. □

Example 2.3.2. *Let $\mathscr{P}\left(\frac{1}{x}\right)$ be a linear functional defined by*

$$
\left(\mathscr{P}\left(\frac{1}{x}\right),\psi\right) = p.v.\int_{\mathbb{R}} \frac{\psi(x)}{x}\,\mathrm{d}x = \lim_{\varepsilon\to0^+}\left[\int_{-\infty}^{-\varepsilon}\frac{\psi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{+\infty}\frac{\psi(x)}{x}\,\mathrm{d}x\right], \quad \psi\in\mathscr{S}(\mathbb{R}).
$$

Then, $\mathscr{P}\left(\frac{1}{x}\right)\in\mathscr{S}'(\mathbb{R})\subset\mathscr{D}'(\mathbb{R})$. Indeed,

$$
\left(\mathscr{P}\left(\frac{1}{x}\right),\psi\right) = p.v.\int_{\mathbb{R}} \frac{\psi(x)}{x}\,\mathrm{d}x = p.v.\int_{\mathbb{R}} \frac{\psi(x)-\psi(0)}{x}\,\mathrm{d}x = \int_{\mathbb{R}} \psi'(\theta x)\,\mathrm{d}x,
$$

since $p.v.\int_{\mathbb{R}} \frac{\psi(0)}{x}\,\mathrm{d}x = 0$, and $\psi(x)-\psi(0) = x\psi'(\theta x)$ for some $\theta\in[0,1]$ by Lagrange's[4] *theorem. Thus,*

$$
\left|\left(\mathscr{P}\left(\frac{1}{x}\right),\psi\right)\right| = \left|\int_{\mathbb{R}}\psi'(\theta x)\,\mathrm{d}x\right| \leqslant \int_{\mathbb{R}} \frac{|\psi'(\theta x)|(1+x^2)}{1+x^2}\,\mathrm{d}x \leqslant \widetilde{q}_{2,1}(\psi)\int_{\mathbb{R}}\frac{\mathrm{d}x}{1+x^2} < +\infty,
$$

(*see Lemma* 2.3.1) *i.e. $\mathscr{P}\left(\frac{1}{x}\right):\mathscr{S}(\mathbb{R})\to\mathbb{C}$ is continuous.*

[3] Baron Augustin-Louis Cauchy (1789–1857) – French mathematician, engineer and physicist.

[4] Joseph-Louis Lagrange (1736–1813) – Italian mathematician, astronomer and physicist (later naturalized French).

2.4 Regular distributions

Regular distributions are defined by locally integrable functions. Therefore, the definition of locally integrable functions is given.

Definition 2.4.1 ([65])**.** *The function f belongs to the space of locally integrable functions on Ω, i.e. $f \in L^{loc}(\Omega)$, if the integral $\int_\Omega f(x)\psi(x)\,\mathrm{d}x$ converges absolutely for every $\psi \in \mathscr{D}(\Omega)$.*

If a function is integrable, then it is also locally integrable. However, the opposite is not true. For example, a non-zero constant function is locally integrable but not integrable on $\mathbb{R}^n$.

For each function $f \in L^{loc}(\Omega)$, the distribution $\widetilde{f}$ is assigned by

$$(\widetilde{f}, \psi) = \int_{\mathbb{R}^n} f(x)\psi(x)\,\mathrm{d}x = \int_K f(x)\psi(x)\,\mathrm{d}x, \quad \psi \in \mathscr{D}(\Omega), \tag{2.4.1}$$

where $K = \operatorname{supp}\psi$.

Definition 2.4.2 ([65])**.** *The distribution defined by a locally integrable function with* (2.4.1) *is called a regular distribution.*

Obviously, two different locally integrable functions define the same distribution if they are equal almost everywhere.

The next theorem shows that the space of locally integrable functions is isomorphic with the subspace of regular distributions.

Theorem 2.4.1 ([65])**.** *Let $f, g \in L^{loc}(\Omega)$ and let $\widetilde{f}$, $\widetilde{g}$ be the corresponding regular distributions. If $(\widetilde{f}, \psi) = (\widetilde{g}, \psi)$ for every $\psi \in \mathscr{D}(\Omega)$, then $f = g$ a.e. in Ω.*

Proof. Let $K = \{x \in \mathbb{R}^n : x_j \in [a_j, b_j],\ j = 1, \dots, n\} \subset \Omega$ be an arbitrary set, and let $\psi(x) = \psi_1(x_1) \cdots \psi_n(x_n)$, where

$$\psi_j(x_j) = \begin{cases} \mathrm{e}^{-1/(x_j - a_j) - 1/(b_j - x_j)}, & x_j \in [a_j, b_j], \\ 0, & \text{otherwise}, \end{cases}$$

for every $j \in \{1, \dots, n\}$. Then, $\psi \in \mathscr{D}(\Omega)$ and $\lim_{\nu \to +\infty} \psi^{1/\nu} = 1$. Since

$$0 = (\widetilde{f}, \psi^{1/\nu}) - (\widetilde{g}, \psi^{1/\nu}) = \int_K \big(f(x) - g(x)\big)\psi^{1/\nu}(x)\,\mathrm{d}x,$$

and $|\psi^{1/\nu}(\cdot)| \leqslant 1$, applying the Lebesgue[5] Dominated Convergence Theorem, it leads to

$$0 = \lim_{\nu \to +\infty} \int_K \big(f(x) - g(x)\big)\psi^{1/\nu}(x)\,\mathrm{d}x = \int_K \big(f(x) - g(x)\big)\,\mathrm{d}x.$$

Therefore, $f = g$ a.e. in Ω. □

Since there is an isomorphism between the spaces $L^{loc}(\Omega)$ and $\mathscr{D}'(\Omega)$, the regular distribution $\widetilde{f}$ defined by $f \in L^{loc}(\Omega)$ will be denoted also by f in the continuation.

However, there exist distributions that are not regular. One of the distributions that is not regular is the Dirac distribution, which is discussed in the next example.

[5]Henri Léon Lebesgue (1875–1941) – French mathematician.

Example 2.4.1. *The Dirac distribution δ_0 is not regular. Indeed, assume that δ_0 is a regular distribution. Then, there is $\delta_0 \in L^{loc}(\Omega)$ such that*

$$(\delta_0, \psi) = \int_{\mathbb{R}^n} \delta_0(x)\psi(x)\,\mathrm{d}x = \psi(0), \quad \psi \in \mathscr{D}(\Omega).$$

Choose $\psi_s(x) = \phi(\frac{x_1}{s}, \dots, \frac{x_n}{s})$, $s > 0$, where ϕ is the function from Example 2.1.1*. Then,*

$$\frac{1}{\mathrm{e}} = \psi_s(0) = \int_{\mathbb{R}^n} \delta_0(x)\psi_s(x)\,\mathrm{d}x = \int_{|x|<s} \delta_0(x)\,\mathrm{e}^{s^2/(|x|^2-s^2)}\,\mathrm{d}x \leqslant \int_{|x|<s} \delta_0(x)\,\mathrm{d}x \to 0,$$

as $s \to 0$, which is impossible. Hence, Dirac distribution δ_0 is not regular.

2.5 Product of generalized functions

The product of two distributions can not be defined in the general case as an operation that is an extension of multiplication of continuous functions. In 1954, Schwartz showed that the product of distributions does not exist over the space whose subspace is the space $\mathscr{D}(\mathbb{R})$. However, it is possible to define the product of a distribution with a smooth function as follows.

Let $f \in L^{loc}(\Omega)$ and $\phi \in \mathscr{C}^\infty(\Omega)$. Then,

$$(\phi f, \psi) = \int_{\mathbb{R}^n} \phi(x) f(x)\psi(x)\,\mathrm{d}x = (f, \phi\psi), \quad \psi \in \mathscr{D}. \tag{2.5.1}$$

The previous equality is taken for the definition of product of $f \in \mathscr{D}'$ and $\phi \in \mathscr{C}^\infty$.

Definition 2.5.1 ([74, 75])**.** *The product of $f \in \mathscr{D}'$ and $\phi \in \mathscr{C}^\infty$ is defined by* (2.5.1)*.*

Since the multiplication of distribution and function $\phi \in \mathscr{C}^\infty$ is linear and continuous mapping between spaces of test functions, the next assertion follows.

Lemma 2.5.1 ([74, 75])**.** *The product of $f \in \mathscr{D}'$ and $\phi \in \mathscr{C}^\infty$ is element of $\mathscr{D}'$ and*

$$\operatorname{supp}(\phi f) \subseteq \operatorname{supp}\phi \cap \operatorname{supp} f.$$

Lemma 2.5.2 ([74, 75])**.** *Let $f \in \mathscr{D}'$. If $\phi \in \mathscr{C}^\infty$ so that $\phi = \mathbf{1}$ in neighborhood of* $\operatorname{supp} f$*, then $f = \phi f$.*

Proof. Let $\psi \in \mathscr{D}$. Then,

$$(f - \phi f, \psi) = ((\mathbf{1} - \phi) f, \psi) = (f, (\mathbf{1} - \phi)\psi) = 0,$$

since $\operatorname{supp} f \cap \operatorname{supp}(\mathbf{1} - \phi)\psi = \emptyset$. Thus, $f - \phi f = \mathbf{0}$, i.e. $f = \phi f$. □

Example 2.5.1. (1) $\phi(\cdot)\delta_0 = \phi(0)\delta_0$*, since* $(\phi\delta_0, \psi) = (\delta_0, \phi\psi) = \phi(0)\psi(0) = (\phi(0)\delta_0, \psi)$*, for every $\psi \in \mathscr{D}$.*

(2) $x\mathscr{P}\left(\frac{1}{x}\right) = 1$*, $x \in \mathbb{R}$, since*

$$\left(x\mathscr{P}\left(\frac{1}{x}\right), \psi\right) = \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = (1, \psi), \quad \psi \in \mathscr{D}.$$

Using Example 2.5.1, it can be see that the product of two distributions can not be defined so that the product is commutative and associative. If it were possible, then it would be

$$\delta_0 = \delta_0 \cdot 1 = \delta_0\left(x\mathscr{P}\left(\frac{1}{x}\right)\right) = (x\delta_0)\cdot\mathscr{P}\left(\frac{1}{x}\right) = 0\cdot\mathscr{P}\left(\frac{1}{x}\right) = \mathbf{0},$$

i.e. $\delta_0 = \mathbf{0}$, which is impossible.

The main reason why it is not possible to extend the product of continuous functions to the product of distributions is that, unlike functions that are defined at each point separately, distributions are defined at a neighborhood of a point, and the value of the distribution at each point is not defined in the general case. The multiplication of distributions can be defined in some cases. If the singular supports of two distributions are disjoint, then their product exists.

Theorem 2.5.1 ([65]). *Let $f, g \in \mathscr{D}'$ so that $\operatorname{sign\,supp} f \cap \operatorname{sign\,supp} g = \emptyset$. If*

$$x_0 \notin \big(\operatorname{sign\,supp} f \cap \operatorname{sign\,supp} g\big),$$

then the distribution h is defined by

$$h(x) = f_{x_0}(x)g(x) \ \text{ or } \ h(x) = g_{x_0}(x)f(x), \quad x \in \mathcal{O}_{x_0},$$

in sense of the definition of the product of a smooth function and a distribution over $\mathcal{O}_{x_0}$, where $\mathcal{O}_{x_0}$ is an open neighborhood of the point x_0 in which f or g is a smooth function, and f_{x_0}, i.e. g_{x_0}, is the restriction of f, i.e. g, over $\mathcal{O}_{x_0}$.

A more detailed overview of different definitions of distribution products is presented in [61, 62].

Mathematicians have dealt with the issue of the product of distributions, because numerous problems in physics, for instance, quantum field theory, are related to the impossibility of defining the product of arbitrary elements from $\mathscr{D}'$. A significant contribution to this problem is the introduction of the product of distributions using the Fourier transform (for more details, see [65]).

Chapter 3

The Fourier transform

Integral transformations play an important role in classical analysis when solving various mathematical models. The theory of generalized functions (theory of distributions) influenced the development of integral transformations since integral transformations are continuous in those spaces. More about integral transformations can be read in [31, 77].

One of the most frequently used integral transformations is the Fourier transform. The theory of the Fourier transform can be found in many books, e.g. [52, 65, 67, 75].

3.1 Fourier transform on $L^2(\mathbb{R}^n)$

As it is already said in Abstract, the Fourier transform is one of the main tools in this research. On the space L^1, it is defined as follows.

Definition 3.1.1 ([40])**.** *The Fourier transform $\mathscr{F}f = \mathscr{F}[f] = \widehat{f}$ of $f \in L^1$ is defined by*

$$\mathscr{F}f(t) = \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-2\pi i\langle x,t\rangle}\, \mathrm{d}x, \quad t \in \mathbb{R}^n. \tag{3.1.1}$$

It immediately follows from Definition 3.1.1 that $\|\widehat{f}\|_{L^\infty} \leqslant \|f\|_{L^1}$. Moreover, Riemann[1]-Lebesque lemma holds.

Lemma 3.1.1 (Riemann-Lebesque, [40])**.** *If $f \in L^1$, then $\widehat{f} \in \mathscr{C}_0$, i.e. $\mathscr{F} : L^1 \to \mathscr{C}_0$.*

The inverse Fourier transform exists under specified conditions. The proof will be given in the next section.

Theorem 3.1.1 ([40])**.** *If $f \in L^1$ and $\widehat{f} \in L^1$, then*

$$\mathscr{F}^{-1}\big[\widehat{f}\,\big](x) = f(x) = \int_{\mathbb{R}^n} \widehat{f}(t)\, \mathrm{e}^{2\pi i\langle x,t\rangle}\, \mathrm{d}t, \quad x \in \mathbb{R}^n. \tag{3.1.2}$$

Remark 3.1.1. *The conditions $f \in L^1$ and $\widehat{f} \in L^1$ imply that $f \in \mathscr{C}_0$. Indeed, applying Lemma* 3.1.1 *to $f \in L^1$ gives $\widehat{f} \in \mathscr{C}_0$ and similarly $\widehat{\widehat{f}} \in \mathscr{C}_0$ follows from condition $\widehat{f} \in L^1$. Using the fact that $\widehat{\widehat{f}}(x) = f(-x)$, $x \in \mathbb{R}^n$, it follows that $f \in \mathscr{C}_0$.*

[1]Georg Friedrich Bernhard Riemann (1826–1866) – German mathematician.

The next statement follows directly from Theorem 3.1.1.

Theorem 3.1.2 ([65])**.** *Let $f \in L^1$ and $\widehat{f}(t) = 0$ for a.e. $t \in \mathbb{R}^n$. Then, $f = \mathbf{0}$.*

In Fourier analysis one of fundamental results is the Plancherel theorem. Namely, Plancherel's theorem proves that the Fourier transform preserves the energy of the signal. This theorem will be used frequently in this dissertation, and the proof will be given in the next section.

Theorem 3.1.3 (**Plancherel**[2]**,** [40])**.** *Let $f \in L^1 \cap L^2$. Then, $\widehat{f} \in L^2$ and*

$$\|f\|_{L^2} = \|\widehat{f}\|_{L^2}. \tag{3.1.3}$$

A consequence of Plancherel's theorem is that Fourier transform can be extended to a unitary operator on L^2 and

$$\langle f, g\rangle_{L^2} = \langle \widehat{f}, \widehat{g}\rangle_{L^2} \quad \text{for all } f, g \in L^2. \tag{3.1.4}$$

Formula (3.1.4) is known as Plancherel's formula. Note, there are other definitions of the Fourier transform (without 2π in the exponent), but then a constant appears in Plancherel's equality (3.1.3) (see [65, 74]).

For arbitrary $f \in L^2$, $\widehat{f}$ can not be defined pointwise with (3.1.1). On L^2 the Fourier transform is defined as follows. Let $Q \subseteq L^1 \cap L^2$ be a dense subspace of L^2, and let $(f_\nu)_{\nu\in\mathbb{N}}$ be such that $f_\nu \in Q$ and $\|f_\nu - f\|_{L^2} \to 0$, as $\nu \to +\infty$. Then, since $f_\nu \in L^1$ for every $\nu \in \mathbb{N}$, it implies that $\widehat{f_\nu}$ is well defined by (3.1.1) for every $\nu \in \mathbb{N}$. The equality (3.1.3) yields that $(\widehat{f_\nu})_{\nu\in\mathbb{N}}$ is a Cauchy sequence in L^2. Since L^2 is a Hilbert space, it follows that $\lim_{\nu\to+\infty} \widehat{f_\nu} = \widehat{f}$.

Moreover, the Fourier transform can be defined on other L^s-spaces.

Theorem 3.1.4 (**Hausdorff**[3]**-Young**[4]**,** [40])**.** *If $s \in [1, 2]$ and r is such that $\frac{1}{s} + \frac{1}{r} = 1$, then the Fourier transform maps L^s into L^r and $\|\widehat{f}\|_{L^r} \leqslant \|f\|_{L^s}$.*

In engineering language, $\mathscr{F}f(t)$ is the amplitude of the frequency t ($\|f\|^2_{L^2}$ is the energy of the signal), while in the physical interpretation, $|\mathscr{F}f(t)|/\|\mathscr{F}f\|^2_{L^2}$ is the probability density for t, where t is the momentum variable.

Example 3.1.1. *The Fourier transform of the function $\psi(x) = \mathrm{e}^{-\pi|x|^2} \in L^1$ is the function $\widehat{\psi}(t) = \mathrm{e}^{-\pi|t|^2}$, $t \in \mathbb{R}^n$. Indeed, it suffices to show the case $n = 1$. Since*

$$\widehat{\psi}(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\pi x^2 - 2\pi i x t}\, \mathrm{d}x = \mathrm{e}^{-\pi t^2} \int_{-\infty}^{+\infty} \mathrm{e}^{-\pi(x+it)^2}\, \mathrm{d}x, \quad t \in \mathbb{R},$$

it is enough to prove that $\int_{-\infty}^{+\infty} \mathrm{e}^{-\pi(x+it)^2}\, \mathrm{d}x = 1$. For $t = 0$ the following calculation shows that the integral is equal to 1,

$$\begin{aligned}\Big(\int_{\mathbb{R}} \mathrm{e}^{-\pi x^2}\, \mathrm{d}x\Big)^2 &= \Big(\int_{\mathbb{R}} \mathrm{e}^{-\pi x^2}\, \mathrm{d}x\Big)\Big(\int_{\mathbb{R}} \mathrm{e}^{-\pi y^2}\, \mathrm{d}y\Big) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-\pi(x^2+y^2)}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_0^{+\infty}\int_0^{2\pi} \rho\, \mathrm{e}^{-\pi\rho^2}\, \mathrm{d}\rho\, \mathrm{d}\theta = 2\pi \lim_{p\to+\infty} \Big(-\frac{\mathrm{e}^{-\pi\rho^2}}{2\pi}\Big)\Big|_{\rho=0}^{\rho=p} = 1,\end{aligned}$$

[2]Michel Plancherel (1885–1967) – Swiss mathematician.
[3]Felix Hausdorff (1868–1942) – German mathematician.
[4]William Henry Young (1863–1942) – English mathematician.

where the change in polar coordinates was used: $x = \rho\cos\theta$, $y = \rho\sin\theta$, $\theta \in [0, 2\pi]$, $\rho > 0$. *The function* $\mathrm{e}^{-\pi(x+it)^2}$, $x, t \in \mathbb{R}$, *is analytic in the complex plane and therefore the integral over the contour from Figure* 2 *of function* $\mathrm{e}^{-\pi(x+it)^2}$ *is equal to zero.*

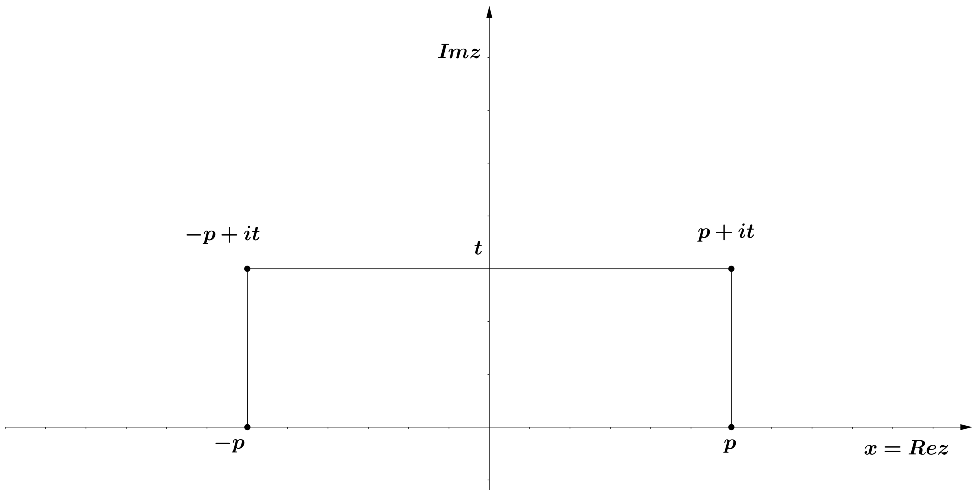

Figure 2.

Thus,

$$\int_{-p}^{p} \mathrm{e}^{-\pi(x+it)^2}\,\mathrm{d}x = \int_{-p}^{p} \mathrm{e}^{-\pi x^2}\,\mathrm{d}x + \int_0^t \mathrm{e}^{-\pi(p+it)^2}\,i\,\mathrm{d}t - \int_0^t \mathrm{e}^{-\pi(-p+it)^2}\,i\,\mathrm{d}t.$$

Since

$$\left|\int_0^t \mathrm{e}^{-\pi(p+it)^2}\,i\,\mathrm{d}t\right| = \left|\int_0^t \mathrm{e}^{-\pi(p^2-t^2)}\,\mathrm{e}^{-2\pi ipt}\,\mathrm{d}t\right| \leqslant |t|\,\mathrm{e}^{-\pi(p^2-t^2)} \to 0$$

and

$$\left|\int_0^t \mathrm{e}^{-\pi(-p+it)^2}\,i\,\mathrm{d}t\right| = \left|\int_0^t \mathrm{e}^{-\pi(p^2-t^2)}\,\mathrm{e}^{2\pi ipt}\,\mathrm{d}t\right| \leqslant |t|\,\mathrm{e}^{-\pi(p^2-t^2)} \to 0,$$

as $p \to +\infty$, *it follows that*

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\pi(x+it)^2}\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-\pi x^2}\,\mathrm{d}x = 1,$$

which had to be proved.

3.2 Basic operators

In this part, basic operators such as translation, modulation, involution and reflection will be defined. Also, the convolution will be defined.

Definition 3.2.1 ([40]). *The translation by $y \in \mathbb{R}^n$ (or the shift by $y \in \mathbb{R}^n$), $T_y f$, is defined by $T_y f(x) = f(x-y)$, $x \in \mathbb{R}^n$. The modulation by $t \in \mathbb{R}^n$, M_t, is defined to be $M_t f(x) = \mathrm{e}^{2\pi i \langle t,x\rangle} f(x)$, $x \in \mathbb{R}^n$.*

In harmonic analysis the fundamental operators take the form $T_y M_t$ and $M_t T_y$, $y, t \in \mathbb{R}^n$. These operators are called time-frequency shifts. It is not difficult to see that

$$T_y M_t = \mathrm{e}^{-2\pi i\langle y,t\rangle} M_t T_y, \quad y,t \in \mathbb{R}^n.$$

Thus, T_y and M_t commute if and only if $\langle y,t\rangle \in \mathbb{Z}^n$. Moreover, these operators are isometric.

Theorem 3.2.1 ([40]). *Let $f \in L^s$, $s \in [1,+\infty]$. Then, $\|T_y M_t f\|_{L^s} = \|f\|_{L^s}$, $y,t \in \mathbb{R}^n$.*

Proof. Let $f \in L^s$, $s \in [1,+\infty)$. Then,

$$\begin{aligned}\|T_y M_t f\|_{L^s}^s &= \int_{\mathbb{R}^n} |T_y M_t f(x)|^s \,\mathrm{d}x = \int_{\mathbb{R}^n} |M_t f(x-y)|^s \,\mathrm{d}x = \int_{\mathbb{R}^n} |f(x-y)|^s \,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} |f(z)|^s \,\mathrm{d}z = \|f\|_{L^s}^s, \quad y,t \in \mathbb{R}^n.\end{aligned}$$

In a similar way, $\|T_y M_t f\|_{L^\infty} = \|f\|_{L^\infty}$ for $f \in L^\infty$, $y,t \in \mathbb{R}^n$. □

Theorem 3.2.2 ([40]). *Let $f \in L^1$. Then, $T_y f, M_y f \in L^1$ and $\widehat{T_y f} = M_{-y}\widehat{f}$, $\widehat{M_y f} = T_y \widehat{f}$, for every $y \in \mathbb{R}^n$.*

Proof. If $f \in L^1$, it is easy to check that $T_y f \in L^1$ and $M_t f \in L^1$, $y,t \in \mathbb{R}^n$. Using Definition 3.1.1,

$$\begin{aligned}\widehat{T_y f}(t) &= \int_{\mathbb{R}^n} (T_y f)(x)\, \mathrm{e}^{-2\pi i\langle x,t\rangle} \,\mathrm{d}x = \int_{\mathbb{R}^n} f(x-y)\, \mathrm{e}^{-2\pi i\langle x,t\rangle} \,\mathrm{d}x \\ &= \mathrm{e}^{-2\pi i\langle y,t\rangle} \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-2\pi i\langle x,t\rangle} \,\mathrm{d}x = M_{-y}\widehat{f}(t), \quad y,t \in \mathbb{R}^n,\end{aligned}$$

and

$$\begin{aligned}\widehat{M_y f}(t) &= \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{2\pi i\langle y,x\rangle}\, \mathrm{e}^{-2\pi i\langle x,t\rangle} \,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-2\pi i\langle x,t-y\rangle} \,\mathrm{d}x \\ &= \widehat{f}(t-y) = T_y \widehat{f}(t), \quad y,t \in \mathbb{R}^n,\end{aligned}$$

which completes the proof. □

Definition 3.2.2 ([40]). *The involution of a function f is the function $f^\diamond$ defined by $f^\diamond(x) = \overline{f(-x)}$, $x \in \mathbb{R}^n$. The reflection operator $\mathscr{I}$ is defined by $\mathscr{I} f(x) = f(-x)$, $x \in \mathbb{R}^n$.*

It can be easily proved that $\widehat{f^\diamond} = \overline{\widehat{f}}$ and $\widehat{\mathscr{I} f} = \mathscr{I}\widehat{f}$. The involution yields $\mathscr{F}^{-1} = \mathscr{I}\mathscr{F}$.

Definition 3.2.3 ([40])**.** *The convolution of functions $f, g \in L^1$, $f * g$, is defined by*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

The fundamental property of convolution is given in the following theorem.

Theorem 3.2.3 ([58])**.** *If $f, g \in L^1$, then:*

(1) $f * g = g * f$,

(2) $\mathscr{F}[f * g] = \mathscr{F}[f]\mathscr{F}[g]$,

(3) $\int_{\mathbb{R}^n} \widehat{f}(t)g(at)\,\mathrm{d}t = \int_{\mathbb{R}^n} f(at)\widehat{g}(t)\,\mathrm{d}t$ *for every $a > 0$.*

Proof. (1) It follows directly from the definition, by substitution of variables.

(2) Using Fubini's[5] theorem (see [63]), it follows that

$$\begin{aligned}\widehat{f * g}(t) &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y\right)\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n} f(y)\,\mathrm{e}^{-2\pi i\langle y,t\rangle}\left(\int_{\mathbb{R}^n} g(x-y)\,\mathrm{e}^{-2\pi i\langle x-y,t\rangle}\,\mathrm{d}x\right)\mathrm{d}y\\ &= \widehat{f}(t)\widehat{g}(t), \quad t \in \mathbb{R}^n.\end{aligned}$$

(3) After a change of variables $\widetilde{t} = at$, $\widetilde{x} = \frac{x}{a}$ and using Fubini's theorem, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(t)g(at)\,\mathrm{d}t &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(x)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}x\right)g(at)\,\mathrm{d}t\\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(a\widetilde{x})\,\mathrm{e}^{-2\pi i\langle \widetilde{x},\widetilde{t}\rangle}\,\mathrm{d}\widetilde{x}\right)g(\widetilde{t})\,\mathrm{d}\widetilde{t}\\ &= \int_{\mathbb{R}^n} f(a\widetilde{x})\left(\int_{\mathbb{R}^n} g(\widetilde{t})\,\mathrm{e}^{-2\pi i\langle \widetilde{x},\widetilde{t}\rangle}\,\mathrm{d}\widetilde{t}\right)\mathrm{d}\widetilde{x}\\ &= \int_{\mathbb{R}^n} f(a\widetilde{x})\widehat{g}(\widetilde{x})\,\mathrm{d}\widetilde{x}, \quad a > 0,\end{aligned}$$

and thus the assertion holds. ☐

Convolution has a particularly important role in the theory of distributions. Differentiation is actually a convolution with the corresponding derivative of δ_0, i.e. $D^a f = D^a\delta_0 * f$, $a \in \mathbb{N}_0^n$; similarly, this is true for translation $T_y f = \delta_y * f$, $y \in \mathbb{R}^n$.

The convolution can be extended to other spaces.

Theorem 3.2.4 (**Young,** [40])**.** *Let $f \in L^p$ and $g \in L^r$. Then, $f * g \in L^s$ and*

$$\|f * g\|_{L^s} \leqslant (A_p A_r A_{s'})^n \|f\|_{L^p}\|g\|_{L^r},$$

where $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{s}$ and $A_p = (\sqrt[p]{p}/\sqrt[p']{p'})^{1/2}$, $p' = \frac{p}{p-1}$.

In the space $L^{loc}(\mathbb{R}^n)$ convolution is introduced as follows.

[5]Guido Fubini (1879–1943) – Italian mathematician.

Definition 3.2.4 ([74, 75]). *The convolution of $f, g \in L^{loc}(\mathbb{R}^n)$ is defined by*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y \quad \text{for a.e. } x \in \mathbb{R}^n,$$

if there exists $\int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y$ for a.e. $x \in \mathbb{R}^n$, and $\int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y \in L^{loc}(\mathbb{R}^n)$.

Since by Definition 3.2.4, $f * g \in L^{loc}(\mathbb{R}^n)$, it is clear that the regular distribution from $\mathscr{D}'$ is determined. Using the Fubini's theorem, it leads to

$$\begin{aligned}\big(f * g, \psi\big) &= \int_{\mathbb{R}^n}\bigg(\int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y\bigg)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(y)\bigg(\int_{\mathbb{R}^n} g(x-y)\psi(x)\,\mathrm{d}x\bigg)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} f(y)\bigg(\int_{\mathbb{R}^n} g(z)\psi(z+y)\,\mathrm{d}z\bigg)\,\mathrm{d}y = \int_{\mathbb{R}^{2n}} f(x)g(y)\psi(x+y)\,\mathrm{d}x\,\mathrm{d}y \qquad (3.2.1)\end{aligned}$$

for every $\psi \in \mathscr{D}$. Since for $\psi(x) \in \mathscr{C}_0^\infty$ the function $\psi(x+y)$, $x, y \in \mathbb{R}^n$, does not have a compact support, the formula (3.2.1) can not define the convolution of arbitrary distributions. Convolution of distributions can be defined as follows.

First, a sequence $(\phi_\nu)_{\nu\in\mathbb{N}} \in \mathscr{C}_0^\infty$ is said to be the unit sequence if both of the following conditions hold:

(1) for every compact set $K \subset \mathbb{R}^n$ there is a $\nu_0(K) \in \mathbb{N}$ so that $\phi_\nu(x) = 1$ for $x \in K$ and $\nu \geqslant \nu_0$;

(2) $(\forall \nu \in \mathbb{N}) \sup\big\{|D^a\phi_\nu(x)| : x \in \mathbb{R}^n\big\} < C_a$, $a \in \mathbb{N}_0^n$.

Such a sequence always exists, for example let $\phi \in \mathscr{C}_0^\infty$ so that $\phi(x) = 1$ for $|x| < 1$, then the sequence $(\phi_\nu)_{\nu\in\mathbb{N}}$ defined by $\phi_\nu(x) = \phi\big(\frac{x}{\nu}\big)$, $\nu \in \mathbb{N}$, $x \in \mathbb{R}^n$, is a unit sequence.

Definition 3.2.5 ([65, 74, 75]). *The convolution $f * g$ of $f, g \in \mathscr{D}'$ can be defined by*

$$\big((f * g)(x), \psi(x)\big) = \lim_{\nu\to+\infty}\big(f(x)g(y), \phi_\nu(x,y)\psi(x+y)\big), \quad \psi \in \mathscr{D},$$

if f and g are such that there exists $\lim_{\nu\to+\infty}\big(f(x)g(y), \phi_\nu(x,y)\psi(x+y)\big)$ for every $\psi \in \mathscr{D}$, where the limes does not depend on choice of the unit sequence $(\phi_\nu(x,y))_{\nu\in\mathbb{N}} \in \mathscr{C}_0^\infty(\mathbb{R}^{2n})$.

According to the given definition, the convolution of distributions does not always exist. There are sufficient conditions for the existence of convolution which will not be stated here (for more details see [65]), but still there are no known conditions for the existence of convolution which would be both necessary and sufficient. With Definition 3.2.5, $f * g$ stays in the distribution space. This property and other properties are given in the next statements.

Theorem 3.2.5 ([74]). *Let $f, g \in \mathscr{D}'$ and $f * g$ exist. Then:*

(1) *$f * g \in \mathscr{D}'$,*

(2) *there exists $g * f$ and $f * g = g * f$,*

(3) *for every $y \in \mathbb{R}^n$ there exists $\big(T_y f * g\big)(x)$, $x \in \mathbb{R}^n$, and*

$$\big(T_y f * g\big)(x) = T_y\big(f * g\big)(x), \quad x \in \mathbb{R}^n.$$

Lemma 3.2.1 ([74]). *If $f \in \mathscr{D}'$, then $f * \delta_0 = \delta_0 * f = f$.*

Similarly, the convolution of Schwartz distributions does not exist in the general case, but it may be defined as follows.

Definition 3.2.6 ([65, 74, 75]). *The tempered convolution $f * g$ of tempered distributions f and g is a linear functional defined by*

$$\big((f*g)(x), \psi(x)\big) = \lim_{\nu\to+\infty} \big(f(x)g(y), \phi_\nu(x,y)\psi(x+y)\big), \quad \psi \in \mathscr{S},$$

if the limes exists for every $\psi \in \mathscr{S}$ and does not depend on the unit sequence $(\phi_\nu(x,y))_{\nu\in\mathbb{N}} \in \mathscr{C}_0^\infty(\mathbb{R}^{2n})$.

It is not difficult to see that also the following statements hold.

Theorem 3.2.6 ([65]). *Let $f, g \in \mathscr{S}'$ and $f * g$ exist. Then:*

(1) *$f * g \in \mathscr{S}'$,*

(2) *there exists $g * f$ and $f * g = g * f$,*

(3) *for every $y \in \mathbb{R}^n$ there exists $\big(T_y f * g\big)(x)$, $x \in \mathbb{R}^n$, and*

$$\big(T_y f * g\big)(x) = T_y\big(f * g\big)(x), \quad x \in \mathbb{R}^n.$$

Lemma 3.2.2 ([65]). *If $f \in \mathscr{S}'$, then $f * \delta_0 = \delta_0 * f = f$.*

Note, a convolution of two distributions always exists if at least one of two distributions has a compact support.

Finally, the proofs of the theorems 3.1.1 and 3.1.3 follows.

Proof of Theorem 3.1.1. Let $g(x) = \mathrm{e}^{-\pi|x|^2}$, $x \in \mathbb{R}^n$. Then, by Example 3.1.1, it follows that $g(0) = \widehat{g}(0) = \int_{\mathbb{R}^n} g(x)\,\mathrm{d}x = 1$. Applying Theorem 3.2.3 (3) and using Example 3.1.1 gives

$$\int_{\mathbb{R}^n} \widehat{f}(t) g(at)\,\mathrm{d}t = \int_{\mathbb{R}^n} f(at)\widehat{g}(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} f(at) g(t)\,\mathrm{d}t, \quad a > 0.$$

Letting $a \to 0$ leads to

$$\int_{\mathbb{R}^n} \widehat{f}(t)\,\mathrm{d}t = f(0) \int_{\mathbb{R}^n} g(t)\,\mathrm{d}t = f(0),$$

since $f \in \mathscr{C}_0$ (see Remark 3.1.1). This is the inverse Fourier transform for $x = 0$. Now, applying $T_{-x}f$ to the last equality and using $\widehat{T_{-x}f} = M_x\widehat{f}$ (by Theorem 3.2.2) gives

$$f(x) = T_{-x}f(0) = \int_{\mathbb{R}^n} \widehat{T_{-x}f}(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} \widehat{f}(t)\,\mathrm{e}^{2\pi i\langle x,t\rangle}\,\mathrm{d}t, \quad x \in \mathbb{R}^n,$$

which completes the proof. □

Proof of Theorem 3.1.3. Let $f \in L^1 \cap L^2$ and $h = f * f^\diamond$. Using Theorem 3.2.3 (2),

$$\widehat{h} = |\widehat{f}|^2 \quad \text{and} \quad h(0) = \int_{\mathbb{R}^n} |f(x)|^2\,\mathrm{d}x,$$

since $\widehat{f^\diamond} = \overline{\widehat{f}}$. On the other hand, from the equality (3.1.2) (for $x = 0$), it follows that

$$h(0) = \int_{\mathbb{R}^n} \widehat{h}(t)\,\mathrm{d}t.$$

Hence, $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$. □

3.3 Fourier transform on $\mathscr{S}(\mathbb{R}^n)$

In the space $\mathscr{S}$, the Fourier transform is defined in the same way as in the space L^1, because $\mathscr{S}$ is dense in L^1 (Theorem 2.3.4).

Definition 3.3.1 ([40]). *The Fourier transform $\mathscr{F}\psi = \mathscr{F}[\psi] = \widehat{\psi}$ of a function $\psi \in \mathscr{S}$ is defined by*

$$\mathscr{F}\psi(t) = \int_{\mathbb{R}^n} M_{-t}\psi(x)\,\mathrm{d}x, \quad t \in \mathbb{R}^n.$$

The Fourier transform is well defined, since from the fact that $\psi \in \mathscr{S}$, it follows that ψ is an absolutely integrable function. Moreover, the Fourier transform is a continuous mapping.

Theorem 3.3.1 ([40, 75]). *Let $\psi \in \mathscr{S}$ and $a \in \mathbb{N}_0^n$.*

(1) $D^a\mathscr{F}[\psi](t) = \mathscr{F}[(-2\pi i x)^a\psi](t)$, $t \in \mathbb{R}^n$.

(2) $\mathscr{F}[D^a\psi](t) = (2\pi i t)^a\mathscr{F}[\psi](t)$, $t \in \mathbb{R}^n$.

(3) $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ *is a continuous linear mapping.*

Proof. (1) Let $\psi \in \mathscr{S}$ and $a \in \mathbb{N}_0^n$. Then, $(-2\pi i x)^a\psi \in \mathscr{S}$ and

$$\begin{aligned} D^a\mathscr{F}[\psi](t) &= D^a \int_{\mathbb{R}^n} \psi(x)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}x = \int_{\mathbb{R}^n} (-2\pi i x)^a\psi(x)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}x \\ &= \mathscr{F}[(-2\pi i x)^a\psi](t), \quad t \in \mathbb{R}^n. \end{aligned}$$

(2) Using integration by parts and the fact that $\psi \in \mathscr{S}$, it follows that

$$\begin{aligned} \mathscr{F}[D^a\psi](t) &= \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi i\langle x,t\rangle}\,D^a\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} (2\pi i t)^a\psi(x)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}x \\ &= (2\pi i t)^a\mathscr{F}[\psi](t), \quad t \in \mathbb{R}^n,\ a \in \mathbb{N}_0^n. \end{aligned}$$

(3) Let $\psi \in \mathscr{S}$ and $a, b \in \mathbb{N}_0^n$. The linearity of the Fourier transform simply follows from the linearity of the integral. By (1) and (2),

$$t^bD^a\mathscr{F}\psi(t) = t^b\mathscr{F}[(-2\pi i x)^a\psi](t) = (-1)^{|a|}(2\pi i)^{|a|-|b|}\mathscr{F}[D^b(x^a\psi)](t), \quad t \in \mathbb{R}^n.$$

Therefore,

$$\begin{aligned} \sup_{t\in\mathbb{R}^n} |t^bD^a\mathscr{F}\psi(t)| &\leqslant (2\pi)^{|a|-|b|} \sup_{t\in\mathbb{R}^n} \int_{\mathbb{R}^n} \left|D^b(x^a\psi)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\right|\mathrm{d}x \\ &= (2\pi)^{|a|-|b|} \int_{\mathbb{R}^n} \left|D^b(x^a\psi)\right|\mathrm{d}x < +\infty, \end{aligned} \tag{3.3.1}$$

since $D^b(x^a\psi) \in \mathscr{S} \subset L^1$. This means that $\mathscr{F}\psi \in \mathscr{S}$. The continuity of the mapping follows from (3.3.1). Indeed, let $\lim_{\nu\to+\infty}\psi_\nu = \psi$ in $\mathscr{S}$. Then, by (3.3.1) for all $a, b \in \mathbb{N}_0^n$,

$$\begin{aligned} \sup_{t\in\mathbb{R}^n} \left|t^bD^a\mathscr{F}[\psi_\nu - \psi](t)\right| &\leqslant (2\pi)^{|a|-|b|} \int_{\mathbb{R}^n} \left|D^b\big(x^a(\psi_\nu - \psi)\big)\right|\mathrm{d}x \\ &\leqslant \sup_{x\in\mathbb{R}^n} \left|D^b\big(x^a(\psi_\nu - \psi)\big)(1+|x|)^{n+1}\right| \int_{\mathbb{R}^n} \frac{(2\pi)^{|a|-|b|}}{(1+|x|)^{n+1}}\,\mathrm{d}x. \end{aligned}$$

Thus, using that $\lim_{\nu\to+\infty}\psi_\nu = \psi$ in $\mathscr{S}$, it yields $\lim_{\nu\to+\infty}\mathscr{F}\psi_\nu = \mathscr{F}\psi$ in $\mathscr{S}$. ☐

Consider the conjugate Fourier transform of $\psi \in \mathscr{S}$, which is defined by

$$\overline{\mathscr{F}}[\psi](t) = \overline{\mathscr{F}}\psi(t) = \int_{\mathbb{R}^n} M_t\psi(x)\,\mathrm{d}x, \quad t \in \mathbb{R}^n.$$

Theorem 3.3.2 ([65])**.** *The conjugate Fourier transform $\overline{\mathscr{F}} : \mathscr{S} \to \mathscr{S}$ is a linear and continuous mapping. Moreover, $\overline{\mathscr{F}}[\mathscr{F}\psi] = \psi$ and $\mathscr{F}[\overline{\mathscr{F}}\psi] = \psi$ for every $\psi \in \mathscr{S}$.*

Proof. The first part of the theorem can be proved in the same way as in Theorem 3.3.1. Let us prove the first equality, the second equality can be proved in a similar way. For $\phi, \psi \in \mathscr{S}$ and $\varepsilon > 0$ hold

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(\varepsilon t)\widehat{\psi}(t)\,\mathrm{e}^{2\pi i\langle x,t\rangle}\,\mathrm{d}t &= \int_{\mathbb{R}^n} \phi(\varepsilon t)\Big[\int_{\mathbb{R}^n} \psi(y)\,\mathrm{e}^{-2\pi i\langle t,y\rangle}\,\mathrm{d}y\Big]\,\mathrm{e}^{2\pi i\langle x,t\rangle}\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \phi(\varepsilon t)\,\mathrm{e}^{-2\pi i\langle y-x,t\rangle}\,\mathrm{d}t\Big]\psi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \varepsilon^{-n}\Big[\int_{\mathbb{R}^n} \phi(t)\,\mathrm{e}^{-2\pi i\langle \frac{y-x}{\varepsilon},t\rangle}\,\mathrm{d}t\Big]\psi(y)\,\mathrm{d}y \\
&= \varepsilon^{-n}\int_{\mathbb{R}^n} \widehat{\phi}\Big(\frac{y-x}{\varepsilon}\Big)\psi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \widehat{\phi}(t)\psi(x+\varepsilon t)\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \qquad (3.3.2)
\end{aligned}$$

by Fubini's theorem and corresponding substitutions. Letting $\varepsilon \to 0^+$, it follows that

$$\phi(0)\int_{\mathbb{R}^n} \widehat{\psi}(t)\,\mathrm{e}^{2\pi i\langle x,t\rangle}\,\mathrm{d}t = \psi(x)\int_{\mathbb{R}^n} \widehat{\phi}(t)\,\mathrm{d}t, \quad x \in \mathbb{R}^n. \qquad (3.3.3)$$

Putting $\widehat{\phi}(t) = \mathrm{e}^{-\pi|t|^2}$, $t \in \mathbb{R}^n$, in equality (3.3.3) yields $\overline{\mathscr{F}}[\mathscr{F}\psi] = \psi$ (by Example 3.1.1). □

Theorem 3.3.3 ([45])**.** *The Fourier transform $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is an isomorphism with the inverse given by*

$$\mathscr{F}^{-1}\big[\widehat{\psi}\big](x) = \psi(x) = \int_{\mathbb{R}^n} M_x\widehat{\psi}(t)\,\mathrm{d}t, \quad x \in \mathbb{R}^n.$$

Proof. According to Theorem 3.3.2, $\overline{\mathscr{F}}[\mathscr{F}\psi] = \psi$ and $\mathscr{F}[\overline{\mathscr{F}}\psi] = \psi$ for every $\psi \in \mathscr{S}$. If $\mathscr{F}\psi = \mathbf{0}$, then $\mathbf{0} = \overline{\mathscr{F}}[\mathscr{F}\psi] = \psi$, i.e. $\mathscr{F}$ is injective. For given $\psi \in \mathscr{S}$, $\mathscr{F}[\overline{\mathscr{F}}\psi] = \psi$ and thus $\mathscr{F}$ is surjective. Hence, $\overline{\mathscr{F}}$ is the inverse of $\mathscr{F}$. Since $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is a continuous linear mapping (by Theorem 3.3.1) and $\overline{\mathscr{F}} : \mathscr{S} \to \mathscr{S}$ is a continuous linear mapping (by Theorem 3.3.2), it follows that $\mathscr{F}$ is an isomorphism. □

The next theorem gives some useful formulas.

Theorem 3.3.4 ([65])**.** *If $\phi, \psi \in \mathscr{S}$, then:*

(1) $\mathscr{F}\big[\mathscr{F}[\psi]\big](x) = \psi(-x)$, $x \in \mathbb{R}^n$,

(2) $\int_{\mathbb{R}^n} \widehat{\psi}(x)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \psi(x)\widehat{\phi}(x)\,\mathrm{d}x$,

(3) $\int_{\mathbb{R}^n} \psi(x)\overline{\phi(x)}\,\mathrm{d}x = \int_{\mathbb{R}^n} \widehat{\psi}(x)\overline{\widehat{\phi}(x)}\,\mathrm{d}x$ *(Parseval's equality)*,

(4) $\int_{\mathbb{R}^n} |\psi(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^n} |\widehat{\psi}(x)|^2\,\mathrm{d}x$.

Proof. It is not difficult to see that (1) holds. Putting $x = 0$ and $\varepsilon = 1$ in (3.3.2), (2) follows. Substituting $\phi(x) = \overline{\psi(x), x \in \mathbb{R}^n}$, into (3) gives (4). It remains to prove (3). Let $\phi_1(x) = \widehat{\phi}(x)$, $x \in \mathbb{R}^n$. Then, $\phi(x) = \widehat{\phi_1}(x)$, $x \in \mathbb{R}^n$. If the mentioned changes are introduced in (2), it leads to (3) for the functions ψ and ϕ_1. □

Theorem 3.3.5 ([65])**.** *The Fourier transform $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ can be extended to L^2 as isometric transform between L^2 spaces, i.e. for $f \in L^2$ the Plancherel's formula* (3.1.3) *holds.*

Proof. Since $\mathscr{S}$ is dense in L^2 (Theorem 2.3.4), by Theorem 3.3.4 (4), the assertion follows. □

The following example is important in the further work.

Example 3.3.1. *Let Δ be the Laplace*[6] *operator, i.e.*

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} = D^{(2,0,\ldots,0,0)} + \cdots + D^{(0,0,\ldots,0,2)}.$$

If $\psi \in \mathscr{S}$, then $\mathscr{F}[\Delta\psi](t) = -4\pi^2|t|^2\widehat{\psi}(t)$, $t \in \mathbb{R}^n$. Indeed, it is sufficient to prove the case $n = 1$. Therefore, by Theorem 3.3.1 (2)*,*

$$\mathscr{F}[\Delta\psi](t) = \mathscr{F}[D^2\psi](t) = (2\pi i t)^2\mathscr{F}[\psi] = -4\pi^2 t^2\mathscr{F}[\psi], \quad t \in \mathbb{R}^n.$$

3.4 Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$

The largest space of distributions making it possible to define the Fourier transform is the space of tempered distributions. Note, if $\psi \in \mathscr{D}(\Omega)$, then $\widehat{\psi} \notin \mathscr{D}(\Omega)$, unless $\psi = \mathbf{0}$.

Let $\varphi \in \mathscr{S}'$, then using Fubini's theorem

$$\begin{aligned}(\widehat{\varphi}, \psi) &= \int_{\mathbb{R}^n} \widehat{\varphi}(t)\psi(t)\,\mathrm{d}t = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \varphi(x)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}x\right)\psi(t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \psi(t)\,\mathrm{e}^{-2\pi i\langle x,t\rangle}\,\mathrm{d}t\right)\varphi(x)\,\mathrm{d}x = (\varphi, \widehat{\psi}), \qquad \psi \in \mathscr{S}.\end{aligned}$$

This equality is taken to define the Fourier transform of a tempered distribution.

Definition 3.4.1 ([75])**.** *The Fourier transform $\mathscr{F}\varphi = \mathscr{F}[\varphi] = \widehat{\varphi}$ of a distribution $\varphi \in \mathscr{S}'$ is defined by*

$$(\mathscr{F}\varphi, \psi) = (\varphi, \mathscr{F}\psi), \quad \psi \in \mathscr{S}.$$

Theorem 3.4.1 ([75])**.** *For every $\varphi \in \mathscr{S}'$ the Fourier transform $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is a continuous mapping with the inverse transform $\mathscr{F}^{-1}[\varphi] = \mathscr{F}^{-1}\varphi$ given by*

$$\mathscr{F}^{-1}[\varphi](x) = \mathscr{F}[\varphi](-x), \qquad x \in \mathbb{R}^n.$$

Proof. Since $\mathscr{F}\psi \in \mathscr{S}$ for every $\psi \in \mathscr{S}$ (by Theorem 3.3.1 (3)), $(\varphi, \mathscr{F}\psi)$ is a functional, obviously linear in $\mathscr{S}$. Let $\lim_{\nu\to+\infty}\psi_\nu = \psi$ in $\mathscr{S}$. Also, using Theorem 3.3.1 (3), it

[6] Pierre-Simon, Marquis de Laplace (1749–1827) – French mathematician and astronomer.

follows that $\lim_{\nu\to+\infty} \mathscr{F}\psi_\nu = \mathscr{F}\psi$ in $\mathscr{S}$, and so $\lim_{\nu\to+\infty}(\varphi, \mathscr{F}\psi_\nu) = (\varphi, \mathscr{F}\psi)$, $\varphi \in \mathscr{S}'$. Thus, the functional $(\varphi, \mathscr{F}\psi)$ is continuous in $\mathscr{S}$. Hence, $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$. Now, let $\lim_{\nu\to+\infty} \varphi_\nu = \varphi$ in $\mathscr{S}'$. Then,

$$\lim_{\nu\to+\infty} (\mathscr{F}\varphi_\nu, \psi) = \lim_{\nu\to+\infty} (\varphi_\nu, \mathscr{F}\psi) = (\varphi, \mathscr{F}\psi) = (\mathscr{F}\varphi, \psi), \quad \psi \in \mathscr{S}.$$

Thus, $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is a continuous mapping. Finally, for every $\psi \in \mathscr{S}$,

$$\begin{aligned}(\mathscr{F}^{-1}\mathscr{F}\varphi, \psi) &= (\mathscr{F}\mathscr{F}\varphi(-x), \psi(x)) = (\mathscr{F}\varphi(-t), \mathscr{F}\psi(t)) = (\mathscr{F}\varphi(t), \mathscr{F}\psi(-t)) \\ &= (\mathscr{F}\varphi(t), \mathscr{F}^{-1}\psi(t)) = (\varphi(x), \mathscr{F}\mathscr{F}^{-1}\psi(x)) = (\varphi, \psi), \quad \varphi \in \mathscr{S}',\end{aligned}$$

i.e. $\mathscr{F}^{-1}\mathscr{F}\varphi = \varphi$. In the same way, it is proved that $\mathscr{F}\mathscr{F}^{-1}\varphi = \varphi$. Thus, $\mathscr{F}^{-1}$ is the inverse of $\mathscr{F}$. □

The next statement is easily proved using Theorem 3.4.1.

Theorem 3.4.2 ([75]). *The Fourier transform $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is an isomorphism.*

Some important properties of the Fourier transform on the space $\mathscr{S}'$ are given in the following statement.

Theorem 3.4.3 ([45, 75]). *If $\varphi \in \mathscr{S}'$ and $a \in \mathbb{N}_0^n$, then:*

(1) $D^a\mathscr{F}[\varphi] = \mathscr{F}[(-2\pi i x)^a\varphi]$, $x \in \mathbb{R}^n$,

(2) $\mathscr{F}[D^a\varphi] = (2\pi i t)^a\mathscr{F}[\varphi]$, $t \in \mathbb{R}^n$,

(3) $\mathscr{F}[T_{x_0}\varphi] = M_{-x_0}\mathscr{F}[\varphi]$, $x_0 \in \mathbb{R}^n$,

(4) $T_{t_0}\mathscr{F}[\varphi] = \mathscr{F}[M_{t_0}\varphi]$, $t_0 \in \mathbb{R}^n$.

Proof. (1) Let $\varphi \in \mathscr{S}'$ and $a \in \mathbb{N}_0^n$. Then, using Definition 2.3.6 and Theorem 3.3.1 (2), it follows that

$$\begin{aligned}(D^a\mathscr{F}[\varphi], \psi) &= (-1)^{|a|}(\mathscr{F}[\varphi], D^a\psi) = (-1)^{|a|}(\varphi, \mathscr{F}[D^a\psi]) = (-1)^{|a|}(\varphi, (2\pi i x)^a\mathscr{F}[\psi]) \\ &= ((-2\pi i x)^a\varphi, \mathscr{F}[\psi]) = (\mathscr{F}[(-2\pi i x)^a\varphi], \psi)\end{aligned}$$

for every $\psi \in \mathscr{S}$, and so $D^a\mathscr{F}[\varphi] = \mathscr{F}[(-2\pi i x)^a\varphi]$, $x \in \mathbb{R}^n$.

(2) Let $\varphi \in \mathscr{S}'$ and $a \in \mathbb{N}_0^n$. Similarly, using Theorem 3.3.1 (1),

$$\begin{aligned}(\mathscr{F}[D^a\varphi], \psi) &= (D^a\varphi, \mathscr{F}[\psi]) = (-1)^{|a|}(\varphi, D^a\mathscr{F}[\psi] = (-1)^{|a|}(\varphi, \mathscr{F}[(-2\pi i t)^a\psi]) \\ &= (\mathscr{F}[\varphi], (2\pi i t)^a\psi) = ((2\pi i t)^a\mathscr{F}[\varphi], \psi), \quad \psi \in \mathscr{S}.\end{aligned}$$

Therefore, $\mathscr{F}[D^a\varphi] = (2\pi i t)^a\mathscr{F}[\varphi]$, $t \in \mathbb{R}^n$.

(3) Let $\varphi \in \mathscr{S}'$ and $x_0 \in \mathbb{R}^n$. Then, by Theorem 3.2.2,

$$\begin{aligned}(\mathscr{F}[T_{x_0}\varphi], \psi) &= (T_{x_0}\varphi, \mathscr{F}[\psi]) = (\varphi, T_{-x_0}\mathscr{F}[\psi]) = (\varphi, \mathscr{F}[M_{-x_0}\psi]) \\ &= (\mathscr{F}[\varphi], M_{-x_0}\psi) = (M_{-x_0}\mathscr{F}[\varphi], \psi), \quad \psi \in \mathscr{S},\end{aligned}$$

and thus $\mathscr{F}[T_{x_0}\varphi] = M_{-x_0}\mathscr{F}[\varphi]$.

(4) The assertion can be proved in a similar way as (3). □

The main concept of classical Fourier analysis is to connected the properties of the function or distribution f with $\widehat{f}$. The smoothness of f implies the decay of $\widehat{f}$, which is stated in the following assertion.

Lemma 3.4.1 ([40]). *Let $a \in \mathbb{N}_0^n$. Then:*

$$D^a f \in L^2 \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 (1+|t|^2)^r \, \mathrm{d}t < +\infty, \quad r \geqslant |a|.$$

Proof. Using Plancherel's theorem and the theorems 2.3.4 and 3.3.1 (2) gives

$$\|D^a f\|_{L^2}^2 = \|\widehat{D^a f}\|_{L^2}^2 = \int_{\mathbb{R}^n} \left|(2\pi i t)^a \widehat{f}(t)\right|^2 \mathrm{d}t = (2\pi)^{2|a|} \int_{\mathbb{R}^n} |t^a|^2 |\widehat{f}(t)|^2 \, \mathrm{d}t, \quad a \in \mathbb{N}_0^n.$$

Let $r \geqslant |a|$. Based on the fact that there exists a constant $C > 0$ so that

$$\frac{1}{C}(1+|t|^2)^r \leqslant \sum_{|a| \leqslant r} |t^a|^2 \leqslant C(1+|t|^2)^r, \quad t \in \mathbb{R}^n,$$

the statement follows. □

The next two examples will be used to prove some of statements that are obtained in this research.

Example 3.4.1. *Let δ_0 be the Dirac distribution (see Example 2.1.2). Since*

$$(\mathscr{F}[\delta_0], \psi) = (\delta_0, \mathscr{F}[\psi]) = \mathscr{F}[\psi](0) = \int_{\mathbb{R}^n} \psi(x) \, \mathrm{d}x = (1, \psi), \quad \psi \in \mathscr{S},$$

it follows that $\mathscr{F}[\delta_0] = 1$. Moreover, $\mathscr{F}[1] = \delta_0$, because $\delta_0 = \mathscr{F}^{-1}[1] = \mathscr{F}[1]$.

Example 3.4.2. *Combining Example 3.4.1 and Theorem 3.4.3 (3) gives*

$$\mathscr{F}[\delta_{x_0}] = \mathscr{F}[T_{x_0}\delta_0] = M_{-x_0}\mathscr{F}[\delta_0] = \mathrm{e}^{-2\pi i \langle x_0, \cdot \rangle}, \quad x_0 \in \mathbb{R}^n.$$

The following example is very important for the next chapter of this dissertation.

Example 3.4.3. *Let $\varphi \in \mathscr{S}'$ and Δ be the Laplace operator (see Example 3.3.1). Then,*

$$\mathscr{F}[(\mathbf{1} - \tfrac{1}{4\pi^2}\Delta)^{r/2}\varphi](t) = (1+|t|^2)^{r/2}\mathscr{F}[\varphi](t), \quad t \in \mathbb{R}^n, \, r \in \mathbb{R}.$$

This follows by applying the binomial formula and by Example 3.3.1.

3.5 Sobolev spaces

The study of Sobolev spaces is of great importance for the theory of partial differential equations. The first results in this area belong to S. L. Sobolev [70, 71]. In this dissertation, only Sobolev spaces which are also Hilbert spaces are considered.

Definition 3.5.1 ([40, 58]). *The Sobolev space (or Bessel*[7] *potential space) $H^r(\mathbb{R}^n)$, $r \in \mathbb{R}$, is defined by*

$$H^r(\mathbb{R}^n) = \left\{ f \in \mathscr{S}' : \mu_r(\cdot)\widehat{f}(\cdot) \in L^2 \right\}.$$

In the continuation, the shorter notation H^r will be used instead of $H^r(\mathbb{R}^n)$. The relationship between the spaces H^r and L^2 is given by the following statement.

Lemma 3.5.1 ([40]). *For every $r > 0$ hold $H^r \subset L^2 \subset H^{-r}$. Moreover, $H^0 = L^2$.*

Proof. The first part of the statement simply follows from the definition of the space H^r. If $r = 0$, then the Plancherel's equality implies that $H^0 = L^2$. □

The Sobolev space H^r, $r \in \mathbb{R}$, is equipped with the inner product

$$\langle f, g \rangle_{H^r} = \int_{\mathbb{R}^n} \widehat{f}(t)\overline{\widehat{g}(t)}\mu_{2r}(t)\,\mathrm{d}t, \tag{3.5.1}$$

and the corresponding norm is

$$\|f\|_{H^r} = \left(\int_{\mathbb{R}^n} |\widehat{f}(t)|^2 \mu_{2r}(t)\,\mathrm{d}t \right)^{1/2}. \tag{3.5.2}$$

Theorem 3.5.1 ([49]). *The space H^r, $r \in \mathbb{R}$, equipped with the inner product* (3.5.1) *is a Hilbert space.*

Proof. It is not difficult to verify that (3.5.1) defines the inner product on the space H^r. The completeness of the space H^r follows by Theorem 3.4.2 and the fact that the space L^2 is complete. □

The connection between the spaces H^r and L^2_r is given in the next statement.

Lemma 3.5.2 ([58]). *Let $r \in \mathbb{R}$. Then, $L^2_r = \mathscr{F}[H^r]$, i.e. $f \in L^2_r$ if and only if $\widehat{f} \in H^r$.*

Proof. Let $r \in \mathbb{R}$. Then,

$$f \in L^2_r \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} |f(t)|^2 \mu_{2r}(t)\,\mathrm{d}t < +\infty \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} \left|\widehat{\widehat{f}}(t)\right|^2 \mu_{2r}(t)\,\mathrm{d}t < +\infty \quad \Leftrightarrow \quad \widehat{f} \in H^r.$$

Therefore, the statement holds. □

Theorem 3.5.2 ([49, 66]). *The space $\mathscr{S}$ is dense in H^r, $r \in \mathbb{R}$.*

Proof. Let $f \in \mathscr{S}$. Then, $\widehat{f} \in \mathscr{S}$ and thus $(1 + |\cdot|^2)^{r/2}\widehat{f}(\cdot) \in \mathscr{S} \subset L^2$, $r \in \mathbb{R}$, by the theorems 3.3.1 (3) and 2.3.4. Hence, $f \in H^r$, $r \in \mathbb{R}$, i.e. $\mathscr{S} \subset H^r$, $r \in \mathbb{R}$. Further, let $r \in \mathbb{R}$ and $f \in H^r$. Since $\mathscr{D}$ is dense in L^2 (by Theorem 2.3.4), it follows that there exists

[7] Friedrich Wilhelm Bessel (1784–1846) – German mathematician, astronomer, physicist and geodesist.

a sequence $(g_\nu)_{\nu\in\mathbb{N}} \in \mathscr{D}$ such that $\lim_{\nu\to+\infty} g_\nu = \widehat{f}\mu_r$ in L^2. Set $f_\nu = \mathscr{F}^{-1}[g_\nu \mu_{-r}]$, $\nu \in \mathbb{N}$. Then, $f_\nu \in \mathscr{S}$, because $g_\nu\mu_{-r} \in \mathscr{D} \subset \mathscr{S}$. Moreover,

$$\begin{aligned}\|f - f_\nu\|^2_{H^r} &= \int_{\mathbb{R}^n} |\widehat{f}(t) - g_\nu(t)\mu_{-r}(t)|^2 \mu_{2r}(t)\,\mathrm{d}t \\ &= \int_{\mathbb{R}^n} |\widehat{f}(t)\mu_r(t) - g_\nu(t)|^2\,\mathrm{d}t \to 0, \quad \text{as } \nu \to +\infty,\end{aligned}$$

which completes the proof. $\square$

Thus, for every $r \in \mathbb{R}$ (see (2.3.7)) hold

$$\mathscr{D} \subset \mathscr{S} \subset H^r \subset \mathscr{S}' \subset \mathscr{D}'. \tag{3.5.3}$$

Remark 3.5.1. *Since $\mathscr{D}$ is dense in $\mathscr{S}$ and $\mathscr{S}$ is dense in H^r, $r \in \mathbb{R}$, it follows that $\mathscr{D}$ is dense in H^r, $r \in \mathbb{R}$, by* (3.5.3)*.*

Some important characteristics of the spaces H^r are given in the following statement.

Theorem 3.5.3 ([35, 38, 49, 58, 65])**.** (1) *If $r, s \in \mathbb{R}$ such that $r \leqslant s$, then $H^s \hookrightarrow H^r$ is continuous.*

(2) *Let $a \in \mathbb{N}_0^n$ and $s \geqslant |a|$. Then, $D^a : H^r \to H^{r-s}$, $r \in \mathbb{R}$, is continuous.*

(3) *If $r \in \mathbb{N}_0$, then $H^r = \{f \in \mathscr{S}' : D^a f \in L^2 \text{ for every } |a| \leqslant r\}$.*

(4) *If $r \in \mathbb{N}_0$ and $f \in H^{-r}$, then $f = \sum_{|a|\leqslant r} D^a f_a$, where $f_a \in L^2$.*

Proof. (1) Let $r \leqslant s$. Then, for $f \in H^s$,

$$\|f\|^2_{H^r} = \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 \mu_{2r}(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} \mu_{2(r-s)}(t)|\widehat{f}(t)|^2\mu_{2s}(t)\,\mathrm{d}t \leqslant \int_{\mathbb{R}^n} |\widehat{f}(t)|^2\mu_{2s}(t)\,\mathrm{d}t = \|f\|^2_{H^s},$$

since $\mu_{2(r-s)}(t) \leqslant 1$, $t \in \mathbb{R}^n$, for $r \leqslant s$.

(2) Let $f \in H^r$. Then, by Theorem 3.4.3 (2),

$$\begin{aligned}\|D^a f\|^2_{H^{r-s}} &= \int_{\mathbb{R}^n} |\widehat{D^a f}(t)|^2 \mu_{2(r-s)}(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} (2\pi)^{2|a|}|t^a|^2|\widehat{f}(t)|^2\mu_{2(r-s)}(t)\,\mathrm{d}t \\ &\leqslant (2\pi)^{2s}\int_{\mathbb{R}^n} |t|^{2s}|\widehat{f}(t)|^2\mu_{2(r-s)}(t)\,\mathrm{d}t \leqslant (2\pi)^{2s}\int_{\mathbb{R}^n} |\widehat{f}(t)|^2\mu_{2r}(t)\,\mathrm{d}t = (2\pi)^{2s}\|f\|_{H^r},\end{aligned}$$

because $|t^a|^2 \leqslant |t|^{2s} \leqslant (1+|t|^2)^s = \mu_{2s}(t)$, $t \in \mathbb{R}^n$, for $s \geqslant |a|$.

(3) The claim follows from Lemma 3.4.1.

(4) Let $f \in H^{-r}$, $r \in \mathbb{N}_0$. Then, $h = \widehat{f}\mu_{-r} \in L^2$ and so

$$\begin{aligned}\widehat{f}(t) = h(t)\mu_r(t) &= \Big(1 + \sum_{j=1}^n |t_j|^r\Big)\frac{h(t)\mu_r(t)}{1 + \sum\limits_{j=1}^n |t_j|^r} = \Big(1 + \sum_{j=1}^n |t_j|^r\Big)g(t) \\ &= g(t) + \sum_{j=1}^n t_j^r \cdot \frac{|t_j|^r}{t_j^r}g(t), \quad t \in \mathbb{R},\end{aligned} \tag{3.5.4}$$

where

$$g(t) = \frac{h(t)\mu_r(t)}{1 + \sum\limits_{j=1}^{n} |t_j|^r} \in L^2.$$

Set $f_0 = \mathscr{F}^{-1} g$ and $f_j = \mathscr{F}^{-1}\big(g|t_j|^r / t_j^r\big)$. Now, applying $\mathscr{F}^{-1}$ to (3.5.4), the assertion follows. $\square$

The following theorem is known as Sobolev's embedding theorem. The proof is more complicated and will be omitted here.

Theorem 3.5.4 ([36, 58])**.** *If $r > \frac{n}{2}$, then $H^r \subset \mathscr{C}_0$. Moreover, if $r > \frac{n}{2} + m$ for some $m \in \mathbb{N}$, then $H^r \subset \mathscr{C}_0^m$.*

Corollary 3.5.1 ([35])**.** *If $f \in H^r$ for every $r \in \mathbb{R}$, then $f \in \mathscr{C}^\infty$.*

The next two theorems are very significant for this dissertation. The first theorem asserts that the multiplication of a function from $\mathscr{S}$ and a distribution from H^r is continuous.

Theorem 3.5.5 ([66])**.** *If $\psi \in \mathscr{S}$ and $f \in H^r$, then $\psi f \in H^r$ and the mapping $f \mapsto \psi f$ is continuous. Moreover, $\|\psi f\|_{H^r} \leqslant C(r,n)\|\widehat{\psi}\mu_{|r|}\|_{L^1}\|f\|_{H^r}$.*

Proof. First, let us prove the estimate

$$\mu_r(t) \leqslant 2^{|r|/2}\mu_r(x)\mu_{|r|}(t-x), \quad r \in \mathbb{R},\ x, t \in \mathbb{R}^n, \tag{3.5.5}$$

known as Peetre's[8] inequality. Since $(1+s)^2 = 1 + 2s + s^2 \leqslant 2(1+s^2)$, $s \in [0, +\infty)$, it implies that

$$\begin{aligned}(1+|t|^2)^{1/2} &\leqslant (1+|x|^2)^{1/2} + |t-x| \leqslant (1+|x|^2)^{1/2}(1+|t-x|)\\ &\leqslant 2^{1/2}(1+|x|^2)^{1/2}(1+|t-x|^2)^{1/2}, \quad x, t \in \mathbb{R}^n.\end{aligned}$$

Hence, the inequality (3.5.5) follows for $r \geqslant 0$. The case $r < 0$ follows from

$$\frac{\mu_r(t)}{\mu_r(x)} = \frac{\mu_{|r|}(x)}{\mu_{|r|}(t)} \leqslant \frac{2^{|r|/2}\mu_{|r|}(t)\mu_{|r|}(t-x)}{\mu_{|r|}(t)} = 2^{|r|/2}\mu_{|r|}(t-x) \quad x, t \in \mathbb{R}^n.$$

Thus, the estimate (3.5.5) holds.

Next, by Theorem 3.2.3 ((1) and (2)) the Fourier transform of ψf is given by

$$\mathscr{F}[\psi f](t) = \int_{\mathbb{R}^n} \widehat{\psi}(t-x)\widehat{f}(x)\,\mathrm{d}x, \quad t \in \mathbb{R}^n.$$

Then,

$$\mathscr{F}[\psi f](t)\mu_r(t) = \int_{\mathbb{R}^n} \widehat{\psi}(t-x)\widehat{f}(x)\frac{\mu_r(t)}{\mu_r(x)}\mu_r(x)\,\mathrm{d}x, \quad t \in \mathbb{R}^n,$$

and by Peetre's inequality (3.5.5),

$$\big|\mathscr{F}[\psi f](t)\mu_r(t)\big| \leqslant C_1 \int_{\mathbb{R}^n} |\widehat{\psi}(t-x)\mu_{|r|}(t-x)| \cdot |\widehat{f}(x)\mu_r(x)|\,\mathrm{d}x = C_1\big(|\widehat{\psi}\mu_{|r|}| * |\widehat{f}\mu_r|\big)(t),$$

for every $t \in \mathbb{R}^n$. Now, by Theorem 3.2.4, it follows that

$$\|\psi f\|_{H^r} = \|\mathscr{F}[\psi f]\mu_r\|_{L^2} \leqslant C_1\big\||\widehat{\psi}\mu_{|r|}| * |\widehat{f}\mu_r|\big\|_{L^2} \leqslant C\|\widehat{\psi}\mu_{|r|}\|_{L^1}\|\widehat{f}\mu_r\|_{L^2} = C\|\widehat{\psi}\mu_{|r|}\|_{L^1}\|f\|_{H^r}.$$

Therefore, the statement holds. $\square$

[8] Jaak Peetre (1935–2019) – Swedish mathematician.

Theorem 3.5.6 ([66])**.** *Let $r, s \in \mathbb{R}$. The mapping $(\mathbf{1} - \frac{1}{4\pi^2}\Delta)^{s/2} : H^{r+s} \to H^r$ defined by*

$$\big(\mathbf{1} - \tfrac{1}{4\pi^2}\Delta\big)^{s/2} f = \mathscr{F}^{-1}[\widehat{f}\mu_s], \quad f \in H^{r+s},$$

is an isometry between spaces H^{r+s} and H^r.

Proof. According to Example 3.4.3, the mapping is well defined and

$$\big\|(\mathbf{1} - \tfrac{1}{4\pi^2}\Delta)^{s/2} f\big\|_{H^r}^2 = \int_{\mathbb{R}^n} |\widehat{f}(t)\mu_s(t)|^2 \mu_{2r}(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 \mu_{2(r+s)}(t)\,\mathrm{d}t = \|f\|_{H^{r+s}}^2$$

for $r, s \in \mathbb{R}$. Therefore, the statement holds. □

Theorem 3.5.7 ([65])**.** *The Sobolev spaces H^r, $r \in \mathbb{R}$, are reflexive and separable.*

Proof. Since L^2 is a reflexive and separable space, from the previous theorem, it follows that H^r, $r \in \mathbb{R}$, are also reflexive and separable spaces. □

Example 3.5.1. *Let δ_0 be the Dirac distribution. Then, $\delta_0 \in H^r$ if and only if $r + \frac{n}{2} < 0$. Indeed, $\mathscr{F}[\delta_0] = 1$ (see Example 3.4.1) yields*

$$\delta_0 \in H^r \quad \Leftrightarrow \quad \mu_r(t)\mathscr{F}[\delta_0](t) \in L^2 \quad \Leftrightarrow \quad \int_{\mathbb{R}^n} (1 + |t|^2)^r\,\mathrm{d}t < +\infty \quad \Leftrightarrow \quad r + \tfrac{n}{2} < 0.$$

In the continuation, it will be proved that the dual space of H^r is the space H^{-r}, $r \in \mathbb{R}$.

Lemma 3.5.3 ([58])**.** *Let $r \geqslant 0$. Then, inner product $\langle \cdot, \cdot \rangle_{L^2} : H^r \times L^2 \to \mathbb{C}$ extends into a continuous sesquilinear[9] form*

$$\langle \cdot, \cdot \rangle_r : H^r \times H^{-r} \to \mathbb{C}, \quad \langle f, g \rangle_r = \int_{\mathbb{R}^n} \widehat{f}(t)\overline{\widehat{g}(t)}\,\mathrm{d}t. \tag{3.5.6}$$

Proof. Let $f \in H^r$ and $g \in L^2$. Then,

$$\big|\langle f, g \rangle_{L^2}\big| = \big|\langle \widehat{f}, \widehat{g} \rangle_{L^2}\big| = \left| \int_{\mathbb{R}^n} \widehat{f}(t)\mu_r(t)\overline{\widehat{g}(t)}\mu_{-r}(t)\,\mathrm{d}t \right| \leqslant \|f\|_{H^r}\|g\|_{H^{-r}},$$

by Plancherel's theorem and Cauchy-Schwarz inequality. Now, the statement follows from Lemma 3.5.1. □

Theorem 3.5.8 ([58])**.** *The form* (3.5.6) *establishes the duality between spaces H^r and H^{-r}, $r \geqslant 0$, i.e.*

$$(H^r)' = \big\{f \mapsto \langle f, g \rangle_r : g \in H^{-r}\big\}, \quad (H^{-r})' = \big\{f \mapsto \langle f, g \rangle_r : g \in H^r\big\}. \tag{3.5.7}$$

Moreover, the isomorphisms $(H^r)' \cong H^{-r}$ and $(H^{-r})' \cong H^r$ are isometries.

Proof. Let $g \in H^{-r}$. Then, $f \mapsto \langle f, g \rangle_r$ is an element of $(H^r)'$, by Lemma 3.5.3. On the other hand, let $\widetilde{f} \in (H^r)'$. Then, by Riesz Representation Theorem, there exists $h \in H^r$ such that $\|h\|_{H^r} = \|\widetilde{f}\|$ and for every $f \in H^r$, $\widetilde{f}(f) = \langle f, h \rangle_{H^r}$. Define $h_1(t) = \widehat{h}(t)\mu_{2r}(t)$, $t \in \mathbb{R}^n$, and let $g = \mathscr{F}^{-1}h_1$. Now, $g \in H^{-r}$ and

$$\widetilde{f}(f) = \langle f, h \rangle_{H^r} = \int_{\mathbb{R}^n} \widehat{f}(t)\overline{\widehat{h}(t)}\mu_{2r}(t)\,\mathrm{d}t = \langle f, g \rangle_r, \quad f \in H^r.$$

The second equality is proved in the same way. Finally, from the previous calculation, it follows that $\|\widetilde{f}\| = \|h\|_{H^r} = \|g\|_{H^{-r}}$. Therefore, the statement is proved. □

[9] A form is sesquilinear if it is linear in the first argument and semi-linear in the second argument.

3.6 The spaces $\mathscr{D}_{L^2}(\mathbb{R}^n)$ and $\mathscr{D}'_{L^2}(\mathbb{R}^n)$

Spaces $\mathscr{D}_{L^2}(\mathbb{R}^n)$ and $\mathscr{D}'_{L^2}(\mathbb{R}^n)$ actually represent the corresponding intersections and the corresponding unions of Sobolev spaces H^r, $r \in \mathbb{R}$, respectively.

Definition 3.6.1 ([65])**.** *The space $\mathscr{D}_{L^2} = \mathscr{D}_{L^2}(\mathbb{R}^n)$ is a subspace of $\mathscr{C}^\infty$ such that $f \in \mathscr{D}_{L^2}$ if and only if $D^a f \in L^2$ for every $a \in \mathbb{N}_0^n$. The topology in $\mathscr{D}_{L^2}$ is defined by the family of norms*

$$\|f\|_m = \Bigg(\sum_{|p| \leqslant m} \|D^p f\|_{L^2}^2 \Bigg)^{1/2}, \quad m \in \mathbb{N}_0.$$

Since the identical mappings from $\mathscr{D}_{L^2}$ to H^r, $r \in \mathbb{N}_0$, are continuous, from Theorem 3.5.3 (3) and Corollary 3.5.1, the next statement holds.

Theorem 3.6.1 ([65])**.** *For the space $\mathscr{D}_{L^2}$ holds $\mathscr{D}_{L^2} = \bigcap_{r=0}^{+\infty} H^r$.*

The dual space of the space $\mathscr{D}_{L^2}$ is denoted by $\mathscr{D}'_{L^2} = \mathscr{D}'_{L^2}(\mathbb{R}^n)$. Using the theorems 3.5.8 and 3.6.1, it follows that the next statement holds.

Theorem 3.6.2 ([65])**.** *$\mathscr{D}'_{L^2} = \bigcup_{r=0}^{+\infty} H^{-r}$.*

Some properties of the spaces $\mathscr{D}_{L^2}$ and $\mathscr{D}'_{L^2}$ are given in the next assertions.

Theorem 3.6.3 ([58, 69])**.** (1) *The space $\mathscr{D}_{L^2}$ is dense in H^r, $r \in \mathbb{R}$.*

(2) *The space $\mathscr{D}$ is dense in $\mathscr{D}_{L^2}$.*

(3) *The space $\mathscr{D}_{L^2}$ is a complete topological vector space, locally convex and reflexive.*

Theorem 3.6.4 ([65, 69])**.** (1) *The space $\mathscr{D}$ is dense in $\mathscr{D}'_{L^2}$.*

(2) *A mapping $D^a : \mathscr{D}'_{L^2} \to \mathscr{D}'_{L^2}$, $a \in \mathbb{N}_0^n$, is continuous.*

(3) *In order for the distribution to belong to $\mathscr{D}'_{L^2}$, it is necessary and sufficient for it to be the finite sum of derivatives of functions from L^2.*

(4) *The distribution $\varphi \in \mathscr{S}'$ if and only if $\varphi = x^a f$ for some $a \in \mathbb{N}_0^n$ and $f \in \mathscr{D}'_{L^2}$.*

Finally, it is not difficult to see that $\mathscr{S} \subset \mathscr{D}_{L^2} \subset \mathscr{D}'_{L^2} \subset \mathscr{S}'$.

Chapter 4

Periodic distributions and wave fronts

The space of periodic distributions is one of basic Schwartz spaces. The motivation for studying periodic distributions stems from local analysis and microanalysis of functions and distributions. In this way, many problems in the scope of $\mathbb{R}^n$ can be simplified and transferred to the torus $\mathbb{T}^n$. More about periodic functions and distributions can be read in [16, 17, 51, 69].

A wave front (or a wave front set) is a term that arose in the period of research related to the classification of singularities using their spectrum and it is at the basis of microlocal analysis. The reader can read more about the wave fronts in [46, 47, 48, 59, 65].

4.1 Periodic functions

Definition 4.1.1 ([17]). *A function $w : \mathbb{R}^n \to \mathbb{C}$ is periodic with period $\eta \in \mathbb{R}^n$, $\eta \neq \mathbf{0}$, if*

$$T_\eta w = w,$$

i.e. $w(x-\eta) = w(x)$, $x \in \mathbb{R}^n$. The set of all continuous periodic functions is denoted by $\mathscr{C}_{pe} = \mathscr{C}_{pe}(\mathbb{R}^n)$, and the set of ℓ-times continuously differentiable periodic functions is denoted by $\mathscr{C}^\ell_{pe} = \mathscr{C}^\ell_{pe}(\mathbb{R}^n)$, $\ell \in \mathbb{N}$.

The norm on the space $\mathscr{C}_{pe}$ is defined by

$$\|w\|_{\mathscr{C}_{pe}} = \sup_{x\in\mathbb{R}^n} |w(x)|, \quad w \in \mathscr{C}_{pe}. \tag{4.1.1}$$

It is not difficult to check that the next statement holds.

Theorem 4.1.1 ([17]). *The space of continuous periodic functions $\mathscr{C}_{pe}$ is a Banach space.*

Example 4.1.1. *The function f defined by*

$$f(x) = \sum_{q\in\mathbb{Z}^n} \alpha_q \, \mathrm{e}^{-2\pi i\langle q,x\rangle}, \quad x \in \mathbb{R}^n,$$

where $(\alpha_q)_{q\in\mathbb{Z}^n}$ is a sequence such that $\sum_{q\in\mathbb{Z}^n} |\alpha_q| < +\infty$, is continuous and periodic.

Definition 4.1.2 ([17]). *A function $w : \mathbb{R}^n \to \mathbb{C}$ is said to be 1-periodic if $T_q w = w$ for every $q \in \mathbb{Z}^n$.*

There is another notation for introduction of periodic functions with value of the n-dimensional torus $\mathbb{T}^n = [-\frac{1}{2}, \frac{1}{2})^n$. An equivalence relation $\sim$ can be introduced on $\mathbb{R}^n$ as follows:

$$x \sim y \quad \text{if and only if} \quad x - y \in \mathbb{Z}^n.$$

The resulting factor space is an n-dimensional torus

$$\mathbb{T}^n = \mathbb{R}^n/_\sim = \mathbb{R}^n/\mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n.$$

Then, 1-periodic functions are identified with their restrictions over $\mathbb{T}^n$ or with their projections on $\mathbb{T}^n$. For example, it is said that w is an element of the space $\mathscr{C}^\ell_{pe}(\mathbb{T}^n)$ if it is periodic (1-periodic) function on $\mathbb{R}^n$ and $w \in \mathscr{C}^\ell_{pe}$.

4.2 The space $\mathscr{P}(\mathbb{R}^n)$

Definition 4.2.1 ([17]). *A subset of $\mathscr{C}_{pe}$ which contains all functions $w \in \mathscr{C}_{pe}(\mathbb{T}^n)$ which are infinitely differentiable, i.e. smooth, is denoted with $\mathscr{P} = \mathscr{P}(\mathbb{R}^n)$, i.e.*

$$\mathscr{P} = \mathscr{P}(\mathbb{R}^n) = \bigcap_{\ell=0}^{+\infty} \mathscr{C}^\ell_{pe}(\mathbb{T}^n), \quad \mathscr{C}^0_{pe}(\mathbb{T}^n) = \mathscr{C}_{pe}(\mathbb{T}^n).$$

The functions from the space $\mathscr{P}$ are called smooth 1-periodic functions.

Example 4.2.1. *The function $f(x) = |\sin x|$ belongs to the space $\mathscr{C}_{pe}$, but does not belong to the space $\mathscr{P}$. Therefore, $\mathscr{P} \neq \mathscr{C}_{pe}$.*

Theorem 4.2.1 ([17]). *The space $\mathscr{P}$ is dense in $\mathscr{C}_{pe}$.*

It is not difficult to notice that if $w \in \mathscr{P}$, then $D^a w \in \mathscr{P}$, $a \in \mathbb{N}_0^n$. Therefore, the topology on $\mathscr{P}$ can be defined by the family of norms

$$\|w\|_{\mathscr{P},\ell} = \sup_{x \in \mathbb{T}^n, |a| \leqslant \ell} |D^a w(x)|, \quad \ell \in \mathbb{N}.$$

A very significant space is the space $L^2(\mathbb{T}^n)$. It contains all measurable periodic square-integrable functions. The space $L^2(\mathbb{T}^n)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{T}^n)} = \int_{\mathbb{T}^n} f(x)\overline{g(x)}\,\mathrm{d}x.$$

Lemma 4.2.1 ([40]). *The set $A_{\mathscr{P}} = \left\{ \mathrm{e}^{-2\pi i \langle q,x \rangle} \in \mathscr{P} : q \in \mathbb{Z}^n,\ x \in \mathbb{T}^n \right\}$ is an orthonormal basis of the space $L^2(\mathbb{T}^n)$.*

A sequence $(w_\nu)_{\nu\in\mathbb{N}} \in \mathscr{P}$ is said to be Cauchy in the space $\mathscr{P}$ if for every $a \in \mathbb{N}_0^n$, $(D^a w_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{C}_{pe}$. Also, a sequence $(w_\nu)_{\nu\in\mathbb{N}} \in \mathscr{P}$ is said to converge to $w \in \mathscr{P}$ if $\lim_{\nu\to+\infty} D^a w_\nu = D^a w$ for every $a \in \mathbb{N}_0^n$.

Theorem 4.2.2 ([17]). *Let $(w_\nu)_{\nu\in\mathbb{N}} \in \mathscr{P}$ be a Cauchy sequence in the space $\mathscr{P}$. Then, it converges to $w \in \mathscr{P}$.*

Proof. It is sufficient to prove the case $n = 1$. Assume that $(w_\nu)_{\nu\in\mathbb{N}} \in \mathscr{P}$ be a Cauchy sequence in the space $\mathscr{P}$. Then, $(D^a w_\nu)_{\nu\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{C}_{pe}$ and thus $\lim_{\nu\to+\infty} D^a w_\nu = u_a \in \mathscr{C}_{pe}$ uniformly for every $a \in \mathbb{N}_0$. Obviously,

$$D^a w_\nu(x) = D^a w_\nu(0) + \int_0^x D^{a+1} w_\nu(t)\,\mathrm{d}t, \quad a \in \mathbb{N}_0.$$

Therefore,

$$\begin{aligned} u_a(x) &= \lim_{\nu\to+\infty} D^a w_\nu(x) = \lim_{\nu\to+\infty} D^a w_\nu(0) + \lim_{\nu\to+\infty} \int_0^x D^{a+1} w_\nu(t)\,\mathrm{d}t \\ &= u_a(0) + \int_0^x u_{a+1}(t)\,\mathrm{d}t, \end{aligned}$$

and thus $Du_a = u_{a+1}$ for every $a \in \mathbb{N}$. This means that if $w = u_0$, then $u_a = D^a w$ and $\lim_{\nu\to+\infty} D^a w_\nu = D^a w$. Hence, $\lim_{\nu\to+\infty} w_\nu = w$ in $\mathscr{P}$. □

The characterization of the space $\mathscr{P}$ is given in the following theorem.

Theorem 4.2.3 ([16])**.** *The function w belongs to the space $\mathscr{P}$ if and only if*

$$w = \sum_{q\in\mathbb{Z}^n} w_q\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle},$$

where $w_q = \int_{\mathbb{T}^n} w(x)\, \mathrm{e}^{-2\pi i\langle q,x\rangle}\,\mathrm{d}x$, $q \in \mathbb{Z}^n$, and $\sum_{q\in\mathbb{Z}^n} |w_q|^2 \mu_{2p}(q) < +\infty$ for every $p \in \mathbb{Z}$.

4.3 The space $\mathscr{P}'(\mathbb{R}^n)$

The dual space of the space $\mathscr{P}$ is $\mathscr{P}' = \mathscr{P}'(\mathbb{R}^n)$.

Definition 4.3.1 ([17])**.** *A continuous linear functional on the space $\mathscr{P}$ is called a periodic distribution. The set of all periodic distributions is denoted by $\mathscr{P}'$.*

If v is a periodic distribution, then v is a tempered distribution, as the following statement says.

Theorem 4.3.1 ([16, 65])**.** *The set of periodic distributions $\mathscr{P}'$ is a subset of the set of tempered distributions $\mathscr{S}'$, i.e. $\mathscr{P}' \subset \mathscr{S}'$.*

Characterization of the space $\mathscr{P}'$ is given in the next theorem.

Theorem 4.3.2 ([16])**.** *The distribution v belongs to the space $\mathscr{P}'$ if and only if*

$$v = \sum_{q\in\mathbb{Z}^n} v_q\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \quad \text{and} \quad \sum_{q\in\mathbb{Z}^n} |v_q|^2 \mu_{-2\tau}(q) < +\infty,$$

for some $\tau > 0$.

To underline the importance of real number τ, $\mathscr{P}'^{\tau}$ is written.

The dual pairing between $v = \sum_{q\in\mathbb{Z}^n} v_q\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \in \mathscr{P}'$ and $w = \sum_{q\in\mathbb{Z}^n} w_q\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \in \mathscr{P}$ is given by

$$(v, w)_{\mathscr{P}',\mathscr{P}} = \sum_{q\in\mathbb{Z}^n} v_q w_q.$$

4.4 Some equality

Some of the more significant equalities used in this dissertation are introduced in this section.

Theorem 4.4.1 (**Plancherel,** [40]). *Let $f \in L^2(\mathbb{T}^n)$ (i.e. $f \in L^2$ be periodic function) and $\widehat{f}(q) = \int_{\mathbb{T}^n} f(t)\, \mathrm{e}^{-2\pi i\langle q,t\rangle}\, \mathrm{d}t$ be the q-th Fourier coefficient. Then,*

$$f = \sum_{q\in\mathbb{Z}^n} \widehat{f}(q)\, \mathrm{e}^{2\pi i\langle q,\cdot\rangle}$$

and

$$\|f\|^2_{L^2(\mathbb{T}^n)} = \int_{\mathbb{T}^n} |f(t)|^2\, \mathrm{d}t = \sum_{q\in\mathbb{Z}^n} |\widehat{f}(q)|^2 = \|(\widehat{f}(q))_{q\in\mathbb{Z}^n}\|^2_{\ell^2}. \tag{4.4.1}$$

Remark 4.4.1. *If $f \in \mathscr{P}$, then the decomposition $f = \sum_{q\in\mathbb{Z}^n} \widehat{f}(q)\, \mathrm{e}^{2\pi i\langle q,\cdot\rangle}$ holds, by Theorem 4.2.3 with $w_{-q} = \widehat{f}(q)$. If additionally $f \in L^2(\mathbb{T}^n)$, then (4.4.1) holds.*

The following "periodization" trick will often be used in the last chapter.

Lemma 4.4.1 ([40]). *Let $f \in L^1$. Then, for every $s > 0$*

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = \int_{[0,s]^n} \left(\sum_{q\in\mathbb{Z}^n} T_{qs} f(x)\right) \mathrm{d}x.$$

Proof. Since $f \in L^1$, by Fubini's theorem, it follows that

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = \sum_{q\in\mathbb{Z}^n} \int_{[0,s]^n+qs} f(x)\, \mathrm{d}x = \int_{[0,s]^n} \left(\sum_{q\in\mathbb{Z}^n} T_{qs} f(x)\right) \mathrm{d}x,$$

Hence, the assertion holds. □

Theorem 4.4.2 (**The Poisson[1] summation formula,** [40]). *Assume that the assumptions $|f| \leqslant C(1+|\cdot|)^{-(n+\varepsilon)}$ and $|\widehat{f}| \leqslant C(1+|\cdot|)^{-(n+\varepsilon)}$, for some $\varepsilon > 0$ and a positive constant C, hold. Then,*

$$\sum_{q\in\mathbb{Z}^n} T_q f(x) = \sum_{q\in\mathbb{Z}^n} M_x \widehat{f}(q), \quad x \in \mathbb{R}^n, \tag{4.4.2}$$

and sums converge absolutely.

Proof. Assume that the conditions given in the theorem hold, and let $h(x) = \sum_{q\in\mathbb{Z}^n} T_q f(x)$. Then, using Lemma 4.4.1,

$$\begin{aligned}\|h\|_{L^1(\mathbb{T}^n)} &= \int_{\mathbb{T}^n} |h(x)|\, \mathrm{d}x = \int_{\mathbb{T}^n} \left|\sum_{q\in\mathbb{Z}^n} T_q f(x)\right| \mathrm{d}x \leqslant \int_{\mathbb{R}^n} |f(x)|\, \mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^n} \frac{C\, \mathrm{d}x}{(1+|x|)^{n+\varepsilon}} < +\infty,\end{aligned}$$

[1] Siméon Denis Poisson (1781–1840) – French mathematician and physicist.

i.e. $h \in L^1(\mathbb{T}^n)$. Thus, using Lemma 4.4.1 again, it follows that

$$\begin{aligned}\widehat{h}(q) &= \int_{\mathbb{T}^n} h(x)\, \mathrm{e}^{-2\pi i\langle q,x\rangle}\, \mathrm{d}x = \int_{\mathbb{T}^n} \Big(\sum_{j\in\mathbb{Z}^n} f(x-j)\, \mathrm{e}^{-2\pi i\langle q,x-j\rangle} \Big)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)\, \mathrm{e}^{-2\pi i\langle q,x\rangle}\, \mathrm{d}x = \widehat{f}(q).\end{aligned}$$

Since

$$\sum_{q\in\mathbb{Z}^n} |\widehat{f}(q)| \leqslant \sum_{q\in\mathbb{Z}^n} \frac{C}{(1+|q|)^{n+\varepsilon}} < +\infty,$$

h has the absolutely convergent Fourier series

$$h(x) = \sum_{q\in\mathbb{Z}^n} M_x \widehat{f}(q), \quad x \in \mathbb{R}^n,$$

which completes the proof. □

Remark 4.4.2. *If*

$$\sum_{q\in\mathbb{Z}^n} T_q f(x) \in L^2(\mathbb{T}^n) \quad \textit{and} \quad \sum_{q\in\mathbb{Z}^n} |\widehat{f}(q)|^2 < +\infty,$$

then holds a weaker version of the Poisson summation formula, i.e. the equality (4.4.2) *holds almost everywhere.*

4.5 Wave fronts

As it is already mentioned in Introduction, the wave front represents a very important mathematical concept in the last fifty years. It is a well-known result that the product of two distributions can be defined if their wave fronts are in the "good" position with respect to each other. This led us to study the product of the observed spaces and wave fronts.

4.5.1 The wave front of distributions

The conic neighborhood of a point is used to define the wave front.

Definition 4.5.1 ([46])**.** *A set* $\Gamma \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ *is called a cone if*

$$t \in \Gamma \quad \textit{implies} \quad \lambda t \in \Gamma \textit{ for every } \lambda > 0.$$

The conic neighborhood of point t_0, *denoted by* Γ_{t_0}, *is an open cone that contains* t_0.

Definition 4.5.2 ([8, 59])**.** (1) *The mapping* $pr_1 : \Omega_1 \times \Omega_2 \to \Omega_1$ *defined by* $pr_1(x,t) = x$ *is called the projection on the first factor.*

(2) *The mapping* $pr_2 : \Omega_1 \times \Omega_2 \to \Omega_2$ *defined by* $pr_2(x,t) = t$ *is called the projection on the second factor.*

The set $\Sigma(\cdot)$ is defined as follows.

Definition 4.5.3 ([46, 48])**.** *It is said that* $t_0 \notin \Sigma(f) \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$*, where* $f \in \mathscr{E}'$*, if there exists a conic neighborhood* Γ_{t_0} *of point* t_0 *such that for every* $t \in \Gamma_{t_0}$ *and for every* $N > 0$*, there exists a constant* $C_N > 0$ *such that*

$$|\widehat{f}(t)| \leqslant C_N \mu_{-2N}(t).$$

It is not difficult to see that $f \in \mathscr{C}_0^\infty$ if and only if $\Sigma(f) = \emptyset$.

Lemma 4.5.1 ([46, 48])**.** *Let* $\phi \in \mathscr{C}_0^\infty$ *and* $f \in \mathscr{E}'$*. Then,* $\Sigma(\phi f) \subseteq \Sigma(f)$*.*

Let $f \in \mathscr{D}'(\Omega)$ and $x \in \Omega$. Set

$$\Sigma_x(f) = \bigcap_{\phi \in \mathscr{C}_0^\infty(\Omega), \phi(x) \neq 0} \Sigma(\phi f).$$

Then, for $\phi \in \mathscr{C}_0^\infty(\Omega)$ such that $\phi(x) \neq 0$, by Lemma 4.5.1, it follows that

$$\lim_{\operatorname{supp} \phi \to \{x\}} \Sigma(\phi f) = \Sigma_x(f).$$

Thus, $\Sigma_x(f) = \emptyset$ if and only if $x \notin \operatorname{sign supp} f$.

Definition 4.5.4 ([46, 48])**.** *The set*

$$WF(f) = \big\{(x, t) \in \Omega \times (\mathbb{R}^n \setminus \{\mathbf{0}\}) : t \in \Sigma_x(f)\big\}$$

is called the wave front set of $f \in \mathscr{D}'(\Omega)$*.*

The definition of a wave front can be reformulated as follows.

Definition 4.5.5 ([46, 48])**.** *The point* $(x_0, t_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ *does not belong to the wave front set* $WF(f)$ *of* $f \in \mathscr{D}'$ *if there exists* $\phi \in \mathscr{C}_0^\infty$ *so that* $\phi(x_0) \neq 0$ *and* $t_0 \notin \Sigma(\phi f)$*.*

Remark 4.5.1. *The statement* $(x, t) \notin WF(f)$ *can be understood as* $f \in \mathscr{C}^\infty$ *at* (x, t)*.*

Obviously, the wave front set is closed in $\Omega \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ and invariant under multiplication by a positive real number of the second factor, i.e. $(x, t) \in WF(f)$ implies $(x, \lambda t) \in WF(f)$ for $\lambda > 0$. Therefore, $WF(f) \subseteq \Omega \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere. Moreover, the wave front set contains all information in $\operatorname{sign supp} f$ and in $\Sigma(f)$ as the following statement says.

Lemma 4.5.2 ([46, 48, 59])**.** *Let* $f \in \mathscr{D}'(\Omega)$*. Then:*

(1) $pr_1(WF(f)) = \operatorname{sign supp} f$,

(2) $pr_2(WF(f)) = \Sigma(f)$.

Proof. (1) Let $x_0 \in \mathbb{R}^n$ so that $x_0 \notin \operatorname{sign supp} f$, and let $\phi \in \mathscr{C}_0^\infty$ satisfy $\operatorname{supp} \phi = \mathcal{K}[x_0, \varepsilon]$, where $\mathcal{K}[x_0, \varepsilon]$ is a sufficiently small closed ball with center at x_0 and radius ε. Then, $\phi f \in \mathscr{C}^\infty$ and ϕf has compact support. Thus, $\phi f \in \mathscr{S}$. Since $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ (by Theorem 3.3.1 (3)), $\widehat{\phi f} \in \mathscr{S}$. Therefore, $(x_0, t_0) \notin WF(f)$.

Conversely, let $x_0 \in \mathbb{R}^n$ so that $(x_0, t_0) \notin WF(f)$. Then, for each $t_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ there are an open set $\mathcal{O}$ containing x_0 and a cone Γ_{t_0}, such that the conditions in Definition 4.5.5 hold. Since the sphere in $\mathbb{R}^n$ is a compact set, it ensures the existence of a finite number of couples $(\mathcal{O}_j, \Gamma_{t_0}^j)$ such that the cones $\Gamma_{t_0}^j$ cover $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For $\phi \in \mathscr{C}_0^\infty$ satisfying $\operatorname{supp} \phi \subseteq \bigcap_j \mathcal{O}_j$ holds $\widehat{\phi f} \in \mathscr{S}$. Therefore, $x_0 \notin \operatorname{sign supp} f$.

(2) According to Definition 4.5.4, it is obvious that $pr_2(WF(f)) \subseteq \Sigma(f)$. In the other direction, let Γ be a conic neighborhood of $pr_2(WF(f))$. Then, for every $x_0 \in \mathbb{R}^n$ there exists a neighborhood $\mathcal{O}_{x_0}$ so that for $\phi \in \mathscr{C}_0^\infty(\mathcal{O}_{x_0})$ holds $\Sigma(\phi f) \subset \Gamma$. Since $\operatorname{supp} f$ is a compact set, there is a finite number of such neighborhoods $\mathcal{O}_{x_j}$. Choose $\phi_j \in \mathscr{C}_0^\infty(\mathcal{O}_{x_j})$ so that $\sum_j \phi_j = 1$ in $\operatorname{supp} f$. Then,

$$\Sigma(f) = \Sigma\Big(\sum_j \phi_j f\Big) \subseteq \bigcup_j \Sigma(\phi_j f) \subset \Gamma.$$

Hence, $\Sigma(f) \subseteq pr_2(WF(f))$. □

Other important properties of wave fronts are given in the next assertions.

Lemma 4.5.3 ([48]). *Let $f \in \mathscr{D}'$. Then:*

(1) $WF(\phi f) \subseteq WF(f)$, $\phi \in \mathscr{C}^\infty$,

(2) $WF(D^a f) \subseteq WF(f)$, $a \in \mathbb{N}_0^n$,

(3) $WF(f+g) \subseteq WF(f) \cup WF(g)$, $g \in \mathscr{D}'$.

Proof. Assertions (1) and (3) follow directly from Definition 4.5.4. To prove (2), let $\psi \in \mathscr{C}_0^\infty$ such that $\psi = \mathbf{1}$ in a neighborhood of x, and let $\widetilde{\psi} \in \mathscr{C}_0^\infty$ so that $\widetilde{\psi} = 1$ in $\operatorname{supp}\psi$. Then,

$$\Sigma_x(D^a f) \subseteq \Sigma(\psi D^a f) = \Sigma(\psi D^a \widetilde{\psi} f) \subseteq \Sigma(D^a \widetilde{\psi} f) \subseteq \Sigma(\widetilde{\psi} f), \quad a \in \mathbb{N}_0^n.$$

Thus, $\Sigma_x(D^a f) \subseteq \lim\limits_{\operatorname{supp}\widetilde{\psi} \to \{x\}} \Sigma(\widetilde{\psi} f) = \Sigma_x(f)$, i.e. the assertion (2) holds. □

Theorem 4.5.1 ([48]). *If $A \subseteq \Omega \times (\mathbb{R}^n \backslash \{\mathbf{0}\})$ is a closed conic, then there exists $f \in \mathscr{D}'(\Omega)$ such that $WF(f) = A$.*

Example 4.5.1. *A wave front for the distribution $\delta_0 \in \mathscr{D}'(\mathbb{R})$ is $WF(\delta_0) = \{(\mathbf{0}, t) : t \neq \mathbf{0}\}$.*

Definition 4.5.6 ([48]). *A distribution $f \in \mathscr{D}'$ is said to be homogenous (of degree s) in $\mathbb{R}^n \setminus \{\mathbf{0}\}$ if*

$$(f, \phi) = c^s (f, \phi_c),$$

where $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$ and $\phi_c(x) = c^n \phi(cx)$, $c > 0$.

Theorem 4.5.2 ([48]). *Let $f \in \mathscr{D}'$ be homogeneous in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Then,*

(1) $(x, t) \in WF(f)$ *if and only if* $(t, -x) \in WF(\widehat{f})$ *for* $t \neq \mathbf{0}$ *and* $x \neq \mathbf{0}$*;*

(2) $x \in \operatorname{supp} f$ *if and only if* $(\mathbf{0}, -x) \in WF(\widehat{f})$ *for* $x \neq \mathbf{0}$*;*

(3) $t \in \operatorname{supp} \widehat{f}$ *if and only if* $(\mathbf{0}, t) \in WF(f)$ *for* $t \neq \mathbf{0}$.

A wave front is also used to determine the existence of the product of two distributions.

Theorem 4.5.3 ([65]). *If $f, g \in \mathscr{D}'(\Omega)$ and*

$$(x, \mathbf{0}) \notin WF(f) \dot{\oplus} WF(g) = \big\{(x, t_1 + t_2) : (x, t_1) \in WF(f),\ (x, t_2) \in WF(g)\big\}, \quad x \in \Omega,$$

then there exists the product fg and

$$WF(fg) \subseteq WF(f) \cup WF(g) \cup \big(WF(f) \dot{\oplus} WF(g)\big).$$

4.5.2 The wave front of Sobolev type

In this research the wave front of Sobolev type will be used. A slightly reformulated Hörmander's definition of Sobolev type wave fronts is given in the following definition.

Definition 4.5.7 ([46, 56]). *It is said that $f \in \mathscr{D}'$ is Sobolev microlocally regular of order $r \in \mathbb{R}$ at $(x_0, t_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ ($f \in H^r_{loc}$ at (x_0, t_0) for short), i.e. $(x_0, t_0) \notin WF_r(f)$, if there is an open cone Γ_{t_0} and $\psi \in \mathscr{D}$, $\psi = \mathbf{1}$ in a neighborhood of x_0 so that*

$$\int_{\Gamma_{t_0}} |\widehat{\psi f}(t)|^2 \mu_{2r}(t)\, \mathrm{d}t < +\infty.$$

Using this definition and some auxiliary statements, Hörmander proves the following statement related to the product of elements of Sobolev spaces.

Theorem 4.5.4 ([46]). *Let $f \in H^r$, $g \in H^s$ and $r + s \geqslant 0$. If $p \leqslant \min\{r, s\}$ and $p \leqslant r + s - \frac{n}{2}$ (if $r = \frac{n}{2}$ or $s = \frac{n}{2}$ or $p = -\frac{n}{2}$, then the inequality is strict), then $fg \in H^p$.*

Theorem 4.5.5 ([46]). *Let $f \in H^r$ and $g \in H^s$.*

1. *If $r > \frac{n}{2}$ and $r + s > \frac{n}{2}$, then $fg \in H^s_{loc}$.*
2. *If $r < \frac{n}{2}$ and $r + s - \frac{n}{2} > p \geqslant 0$, then $fg \in H^p_{loc}$.*
3. *If $r + s > 0$, then $fg \in H^{r+s-\frac{n}{2}}_{loc}$.*

Let $\theta \in (0, 1]$. The set

$$\mathbb{T}^n_{x,\theta} = \prod_{j=1}^{n} \left(x_j - \tfrac{\theta}{2}, x_j + \tfrac{\theta}{2}\right)$$

is denoted by $\mathbb{T}^n_{x,\theta}$.

Definition 4.5.8 ([56]). *If $f \in \mathscr{D}'$ has the support in $\mathbb{T}^n_{x_0,\theta}$, $\theta \in (0, 1)$, the periodic extension of localization of f in some neighborhood of the point x_0 is*

$$f_{pe}(x) = \sum_{q \in \mathbb{Z}^n} T_q f(x).$$

By discretization of the wave front, the authors in [56] concluded the following statement.

Theorem 4.5.6 ([56]). *Let $f \in \mathscr{D}'$. The following conditions are equivalent.*

1. *There is an open cone Γ_{t_0} and $\psi \in \mathscr{D}(\mathbb{T}^n_{x_0,\theta})$, $\psi = \mathbf{1}$ in a neighborhood of x_0, $\theta \in (0, 1)$, so that*

$$\sum_{q \in \mathbb{Z}^n \cap \Gamma_{t_0}} |\alpha_q|^2 \mu_{2r}(q) < +\infty, \quad \text{where } (\psi f)_{pe} = \sum_{q \in \mathbb{Z}^n} \alpha_q \, \mathrm{e}^{-2\pi i \langle q, \cdot \rangle}.$$

2. *$(x_0, t_0) \notin WF_r(f)$.*

Proof. Suppose that condition (1) holds. Let Γ be an open cone such that $t_0 \in \Gamma$ and $\overline{\Gamma} \subset \Gamma_{t_0} \cup \{\mathbf{0}\}$. Choose $\varepsilon \in (0, \theta)$ so that $\psi = \mathbf{1}$ in $\mathbb{T}^n_{x_0,\varepsilon}$. First, it is necessary to prove the next assertion: if $Q \subset \mathscr{D}'(\mathbb{T}^n_{x_0,\varepsilon})$ is a bounded set, then

$$\sup_{\phi \in Q} \sum_{q \in \Gamma \cap \mathbb{Z}^n} \left|\widehat{\phi f}(q)\right|^2 \mu_{2r}(q) < +\infty. \tag{4.5.1}$$

Let $Q \subset \mathscr{D}'(\mathbb{T}^n_{x_0,\varepsilon})$ be a fixed bounded set. Since $\phi f = \psi\phi f$, it gives

$$\widehat{\phi f}(q) = \sum_{p\in\mathbb{Z}^n} \alpha_q \widehat{\phi}(q-p) \quad \text{for every } \phi \in Q.$$

Fix a constant $c \in (0,1)$ such that $c < \min\{d(\partial\Gamma_{t_0}, \overline{\Gamma}\cap\mathbb{S}^{n-1}), d(\partial\Gamma, \overline{\mathbb{R}^n\setminus\Gamma_{t_0}}\cap\mathbb{S}^{n-1})\}$, where d is the distance between two sets. Then, $|t_1 - t_2| > c\max\{|t_1|,|t_2|\}$, $t_1 \in \Gamma$, $t_2 \in \Gamma_{t_0}$. Using Peetre's inequality (3.5.5), it follows that

$$\begin{aligned}\Big(\sum_{q\in\Gamma\cap\mathbb{Z}^n} \big|\widehat{\phi f}(q)\big|^2\Big)^{1/2} &\leqslant C\bigg(\sum_{q\in\Gamma\cap\mathbb{Z}^n}\Big(\sum_{p\in\mathbb{Z}^n}|\alpha_p|\mu_r(p)|\widehat{\phi}(q-p)|\mu_{|r|}(q-p)\Big)^2\bigg)^{1/2}\\ &\leqslant C\big(I_1(\phi)+I_2(\phi)\big),\end{aligned}$$

where $I_1(\phi)$ and $I_2(\phi)$ are

$$I_1(\phi) = \bigg(\sum_{q\in\Gamma\cap\mathbb{Z}^n}\Big(\sum_{p\in\Gamma_{t_0}\cap\mathbb{Z}^n}|\alpha_p|\mu_r(p)|\widehat{\phi}(q-p)|\mu_{|r|}(q-p)\Big)^2\bigg)^{1/2},$$

$$I_2(\phi) = \bigg(\sum_{q\in\Gamma\cap\mathbb{Z}^n}\Big(\sum_{p\notin\Gamma_{t_0}\cap\mathbb{Z}^n}|\alpha_p|\mu_r(p)|\widehat{\phi}(q-p)|\mu_{|r|}(q-p)\Big)^2\bigg)^{1/2}.$$

Young's inequality leads to

$$\sup_{\phi\in Q} I_1(\phi) \leqslant \Big(\sum_{q\in\Gamma_{t_0}\cap\mathbb{Z}^n}|\alpha_q|^2\mu_{2r}(q)\Big)^{1/2}\sup_{\phi\in Q}\sum_{q\in\mathbb{Z}^n}|\phi(q)|\mu_{|r|}(q) < +\infty,$$

since Q is a bounded set. Further, for the estimate $I_2(\phi)$, the next two estimates are used:

(a) for every $q \in \mathbb{Z}^n$, $|\alpha_q|^2\mu_r(q) = |\widehat{\psi f}(q)|\mu_r(q) \leqslant C_1\mu_s(q)$ for some $C_1 > 0$ and $s > 0$;

(b) $|\widehat{\phi}(q)| \leqslant C_2\mu^{-1}_{s+|r|+3(n+1)/2}(q)$ for some $C_2 > 0$.

The estimate (a) follows from the fact that ψf has a compact support, while (b) follows from the fact that Q is bounded. Thus,

$$\begin{aligned}\sup_{\phi\in Q}\big(I_2(\phi)\big)^2 &\leqslant C\sum_{q\in\Gamma\cap\mathbb{Z}^n}\Big(\sum_{p\notin\Gamma_{t_0}\cap\mathbb{Z}^n}\mu_s(p)\mu^{-1}_{s+3(n+1)/2}(q-p)\Big)^2\\ &\leqslant Cc^{-2s-3(n+1)}\sum_{q\in\Gamma\cap\mathbb{Z}^n}\mu^{-1}_{n+1}(q)\Big(\sum_{p\notin\Gamma_{t_0}\cap\mathbb{Z}^n}\mu^{-1}_{n+1}(p)\Big)^2.\end{aligned}$$

Hence, (4.5.1) holds.

Let us choose an open cone Γ_1 so that $t_0 \in \Gamma_1$ and $\Gamma_1 \subset \Gamma\cup\{\mathbf{0}\}$. Let $\varpi \in \mathscr{D}(\mathbb{T}^n_{x_0,\varepsilon})$ so that $\varpi = \mathbf{1}$ in a neighborhood of x_0. Choose $R > 0$ so that $\Gamma_1\cap\{t\in\mathbb{R}^n : |t| \geqslant R\} \subset (\Gamma\cap\mathbb{Z}^n)+[0,1)^n$. Set $Z_q = q + [0,1]^n$, $q \in \Gamma\cap\mathbb{Z}^n$. Then,

$$\int_{\substack{t\in\Gamma_1\\|t|\geqslant R}}\big|\mathscr{F}[\varpi f](t)\big|^2\mu_{2r}(q)\,\mathrm{d}t \leqslant C\sum_{q\in\Gamma\cap\mathbb{Z}^n}\mu_{2r}(q)\int_{Z_q}\big|\mathscr{F}[\varpi f](t)\big|^2\,\mathrm{d}t$$

$$= C \int_{[0,1]^n} \sum_{q \in \Gamma \cap \mathbb{Z}^n} \left| \mathscr{F}[\varpi f](q+x) \right|^2 \mu_{2r}(q) \,\mathrm{d}x$$
$$\leqslant C \sup_{x \in [0,1]^n} \sum_{q \in \Gamma \cap \mathbb{Z}^n} \left| \mathscr{F}[M_{-x} \varpi f](q) \right|^2 \mu_{2r}(q) < +\infty,$$

by Peetre's inequality (3.5.5) and (4.5.1). Thus, $(x_0, t_0) \notin WF_r(f)$, i.e. (2) holds.

Suppose that $(x_0, t_0) \notin WF_r(f)$, i.e. that the condition (2) holds. Then, there are $\varepsilon \in (0,1)$ and an open cone Γ_{x_0} so that

$$\sup_{\varpi \in Q} \int_{\Gamma_{x_0}} |\widehat{\varpi f}(t)|^2 \,\mathrm{d}t < +\infty \tag{4.5.2}$$

for every bounded set $Q \subset \mathscr{D}(\mathbb{T}^n_{x_0,\varepsilon})$. This claim can be proved by the analysis similar to that in the proof of (4.5.1). Further, let Γ be an open cone so that $t_0 \in \Gamma$ and $\overline{\Gamma} \subset \Gamma_{x_0} \cup \{\mathbf{0}\}$. Then, there are $R > 0$ so that $(\Gamma + [0,1]^n) \cap \{t \in \mathbb{R}^n : |t| \geqslant R\} \subset \Gamma_{x_0}$. Let $\psi \in \mathscr{D}(\mathbb{T}^n_{\varepsilon,x_0})$ so that $\psi = \mathbf{1}$ in a neighborhood of x_0, and let $h : \Gamma_{x_0} \to [0,1]^n$ be a measurable function. Consider the bounded set

$$Q = \left\{ \varpi_{j,h} \in \mathscr{D}(\mathbb{T}^n_{x_0,\varepsilon}) : \varpi_{j,h}(x) = x_j \,\mathrm{e}^{-2\pi i \langle x, h(t) \rangle} \,\psi(x),\, t \in \Gamma_{x_0},\, j = 1, \ldots, n \right\}$$

in (4.5.2). Then, there exists a constant $C > 0$ (C is independended of h) so that

$$\int_{\Gamma_{x_0}} \left| \nabla \mathscr{F}[\psi f](t + h(t)) \right|^2 \mu_{2r}(t) \,\mathrm{d}t < C, \tag{4.5.3}$$

where ∇ is the operator nabla, i.e. $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. Let $Z_q = q + [0,1]^n$, $q \in \mathbb{Z}^n$. Note, $Z_q \subset \Gamma_{x_0}$ for $|q| \geqslant R$. Therefore,

$$\left(\sum_{q \in \Gamma \cap \mathbb{Z}^n} |\widehat{\psi f}(q)|^2 \mu_{2r}(q) \right)^{1/2} = \left(\sum_{q \in \Gamma \cap \mathbb{Z}^n} \int_{Z_q} |\widehat{\psi f}(q)|^2 \mu_{2r}(q) \,\mathrm{d}t \right)^{1/2} \leqslant I_1 + I_2,$$

where $I_1^2 = \sum_{q \in \Gamma \cap \mathbb{Z}^n} \int_{Z_q} |\widehat{\psi f}(q) - \widehat{\psi f}(t)|^2 \mu_{2r}(q) \,\mathrm{d}t$, and

$$I_2^2 = \sum_{q \in \Gamma \cap \mathbb{Z}^n} \int_{Z_q} |\widehat{\psi f}(t)|^2 \mu_{2r}(q) \,\mathrm{d}t$$
$$\leqslant \sum_{|q| \leqslant R} \int_{Z_q} |\widehat{\psi f}(t)|^2 \mu_{2r}(q) \,\mathrm{d}t + C_1 \int_{\Gamma_{x_0}} |\widehat{\psi f}(t)|^2 \mu_{2r}(t) \,\mathrm{d}t < +\infty.$$

It remains to prove that $I_1 < +\infty$. For given $z > 0$ define $h_z : \Gamma_{x_0} \to [0,1]^n$ by

$$h_z(t) = \begin{cases} z(q - t), & t \in Z_q,\ |t| \geqslant R, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$|\widehat{\psi f}(t) - \widehat{\psi f}(q)|^2 \leqslant |q - t| \int_0^1 \left| \nabla \mathscr{F}[\psi f](t + z(q - t)) \right|^2 \mathrm{d}z,$$

it follows that

$$I_1^2 \leqslant \sum_{\substack{|q| \leqslant R \\ q \in \Gamma \cap \mathbb{Z}^n}} \int_{Z_q} |\widehat{\psi f}(q) - \widehat{\psi f}(t)|^2 \mu_{2r}(q) \,\mathrm{d}t + C' \sup_{z \in [0,1]} \int_{\Gamma_{x_0}} \left| \nabla \mathscr{F}[\psi f](t + h_z(t)) \right|^2 \mu_{2r}(t) \,\mathrm{d}t$$
$$< +\infty.$$

Therefore, (1) holds. □

4.6 The spaces $\mathscr{P}^{s,r}$ and $\mathscr{P}^{s,r}_{loc}$

In this research, the spaces $\mathscr{P}^{1,r}$, $\mathscr{P}^{2,r}$ and $\mathscr{P}^{1,r}_{loc}$, $\mathscr{P}^{2,r}_{loc}$ are used to determine the appropriate products. Therefore, the spaces $\mathscr{P}^{s,r}$ and $\mathscr{P}^{s,r}_{loc}$ are introduced in this part.

Definition 4.6.1 ([16]). *The spaces $\mathscr{P}^{s,r}$, $s \geqslant 1$, $r \in \mathbb{R}$, are defined by*

$$\mathscr{P}^{s,r} = \Big\{ f \in \mathscr{P}' : f = \sum_{q\in\mathbb{Z}^n} \alpha_q \,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle},\ (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell^s_r \Big\},$$

with the corresponding norm $\|f\|_{\mathscr{P}^{s,r}} = \|(\alpha_q)_{q\in\mathbb{Z}^n}\|_{\ell^s_r}$.

Lemma 4.6.1 ([16]). *The spaces $\mathscr{P}^{s,r}$, $s \geqslant 1$, $r \in \mathbb{R}$, are Banach spaces.*

Definition 4.6.2 ([16]). *The function $(\phi f)_{pe}$ of $f \in \mathscr{D}'$ and $\phi \in \mathscr{D}(\mathbb{T}^n_{x_0,\theta})$ is defined by*

$$(\phi f)_{pe} = \sum_{q\in\mathbb{Z}^n} \alpha_q \,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle},$$

where $\alpha_q = \int_{\mathbb{T}^n_{x_0,\theta}} (\phi f)(t)\,\mathrm{e}^{-2\pi i\langle q,t\rangle}\,\mathrm{d}t$, $q \in \mathbb{Z}^n$.

Definition 4.6.3 ([16]). *The local spaces $\mathscr{P}^{s,r}_{loc}$, $s \geqslant 1$, $r \in \mathbb{R}$, are defined by*

$$\mathscr{P}^{s,r}_{loc} = \Big\{ f \in \mathscr{D}' : (\phi f)_{pe} \in \mathscr{P}^{s,r} \text{ for all } x_0 \in \mathbb{R}^n \text{ and } \phi \in \mathscr{D}(\mathbb{T}^n_{x_0,\theta}) \Big\}.$$

The topology in the local spaces $\mathscr{P}^{s,r}_{loc}$, $s \geqslant 1$, $r \in \mathbb{R}$, is defined by the family of seminorms

$$\|f\|_{x_0,\phi} = \|(\phi f)_{pe}\|_{\mathscr{P}^{s,r}}.$$

Lemma 4.6.2 ([56]). *For all $s \geqslant 1$ and $r \in \mathbb{R}$ hold:*

(1) $\mathscr{P}^{s,r} \subseteq \mathscr{P}^{s,r}_{loc}$,

(2) $\mathscr{P} = \bigcap\limits_{r\geqslant 0} \mathscr{P}^{s,r}$,

(3) $\mathscr{P}' = \bigcup\limits_{r\leqslant 0} \mathscr{P}^{s,r}$.

The product of two distributions from spaces $\mathscr{P}^{s_1,r}$ and $\mathscr{P}^{s_2,r}$ is defined by Fourier coefficients.

Definition 4.6.4 ([56]). *The product of functions $f_1 = \sum_{q\in\mathbb{Z}^n} f_{q,1}\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \in \mathscr{P}^{s_1,r}$ and $f_2 = \sum_{q\in\mathbb{Z}^n} f_{q,2}\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \in \mathscr{P}^{s_2,r}$ is defined by*

$$f = f_1 f_2 = \sum_{q\in\mathbb{Z}^n} f_q \,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle},$$

where $f_q = \sum_{p\in\mathbb{Z}^n} f_{q-p,1} f_{p,2}$, $q \in \mathbb{Z}^n$.

Theorem 4.6.1 ([56]). *Let $f_1 \in \mathscr{P}^{s_1,r}$ and $f_2 \in \mathscr{P}^{s_2,r}$. If $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s} + 1$, then the mapping*

$$f_1 f_2 : \mathscr{P}^{s_1,r} \times \mathscr{P}^{s_2,r} \to \mathscr{P}^{s,r} \tag{4.6.1}$$

is continuous. Moreover, if $f_1 \in \mathscr{P}^{s_1,r_1}$, $f_2 \in \mathscr{P}^{s_2,r_2}$ *and* $r, r_1, r_2 \in \mathbb{R}$ *such that* $r_1 + r_2 \geqslant 0$ *and* $r \leqslant \min\{r_1, r_2\}$, *then the mapping*

$$f_1 f_2 : \mathscr{P}^{s_1,r_1} \times \mathscr{P}^{s_2,r_2} \to \mathscr{P}^{s,r} \tag{4.6.2}$$

is also continuous, where $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s} + 1$.

Proof. By Young's inequality, it follows that $\|f_1 f_2\|_{\mathscr{P}^{s,r}} \leqslant C\|f_1\|_{\mathscr{P}^{s_1,r}}\|f_2\|_{\mathscr{P}^{s_2,r}}$. Therefore, the mapping (4.6.1) is continuous. Further, assume that $r_1 \geqslant 0$ and $r = r_2$. Then, $r_1 \geqslant |r_2|$ provided that $r_1 + r_2 \geqslant 0$. Now, using Peetre's inequality (3.5.5), it follows that the mapping (4.6.2) is continuous. □

The product in local versions of these spaces is introduced as follows.

Definition 4.6.5 ([56]). *Let* $f_1 \in \mathscr{P}^{s_1,r_1}_{loc}$, $f_2 \in \mathscr{P}^{s_2,r_2}_{loc}$, $\theta \in (0,1)$, *and let* $\phi \in \mathscr{D}(\mathbb{T}^n_{x_0,1})$ *be so that* $\phi(x) = 1$ *for all* $x \in \mathbb{T}^n_{x_0,\varepsilon}$, $\varepsilon < \theta$. *The product* $f = f_1 f_2$ *is defined locally by* $f_{x_0,\theta} \in \mathscr{D}'(\mathbb{T}^n_{x_0,\theta})$, *where* $f_{x_0,\theta}$ *is the restriction of the product* $(\phi f_1)_{pe}(\phi f_2)_{pe}$ *to* $\mathbb{T}^n_{x_0,\theta}$.

The following statement is a consequence of Theorem 4.6.1.

Corollary 4.6.1 ([56]). *Let* $f_1 \in \mathscr{P}^{s_1,r}_{loc}$ *and* $f_2 \in \mathscr{P}^{s_2,r}_{loc}$. *If* $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s} + 1$, *then the mapping*

$$f_1 f_2 : \mathscr{P}^{s_1,r}_{loc} \times \mathscr{P}^{s_2,r}_{loc} \to \mathscr{P}^{s,r}_{loc}$$

is continuous. Moreover, if $f_1 \in \mathscr{P}^{s_1,r_1}_{loc}$, $f_2 \in \mathscr{P}^{s_2,r_2}_{loc}$ *and* $r, r_1, r_2 \in \mathbb{R}$ *such that* $r_1 + r_2 \geqslant 0$ *and* $r \leqslant \min\{r_1, r_2\}$, *then the mapping*

$$f_1 f_2 : \mathscr{P}^{s_1,r_1}_{loc} \times \mathscr{P}^{s_2,r_1}_{loc} \to \mathscr{P}^{s,r}_{loc}$$

is also continuous, where $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s} + 1$.

Chapter 5

Frame theory in Hilbert spaces

The theory of frames belongs to the branch of modern mathematics and it has seen rapid development in the last twenty years. This was primarily contributed by a wide field of applications, primarily in signal analysis. As an advantage of using frames in various algorithms, they state efficiency, compression of data, speed of numerical calculations, and removal of noise. When transmitting data over the internet, the decomposition coefficients of a signal and the corresponding tools of linear algebra and numerical mathematics are used to create fast and reliable algorithms that decompose, process, transmit, store, and reconstruct the given signal. The orthogonality condition makes it impossible to reconstruct the lost coefficients from the obtained ones so that part of the information they carry is lost forever. When transmitting an image or sound, it turns out that algorithms based on results of linear algebra, numerical analysis, and operator theory becomes more efficient if the uniqueness condition is omitted. In this way, the most important properties of orthonormal bases, linear independence and orthogonality, lead to serious difficulties. On the other hand, frameworks can be constructed to meet certain specificities imposed by nature of problem. Today, frame theory is used to compress fingerprint images that make up the FBI files, to remove noise from audio signals, to remove white noise from satellite photos, determine the level of different layers of the earth based on the reflection of acoustic waves emitted from the surface, etc.

Frames are a more flexible tool than an orthonormal basis, because they allow each vector in a vector space equipped with an inner product can be written as a linear combination of the elements in a frame, but linear independence and orthogonality are not required between the frame elements.

In this chapter, only frames in Hilbert spaces are presented. However, K. Gröchening extended the theory of frames to a large class of general Banach spaces, but as the theory there is somewhat more complex and is not needed by us, it will be omitted in this chapter. For a more complete study of frame theory, the reader can see [28, 29, 30, 40, 41, 44].

5.1 Basic terms and definitions

In this dissertation, $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ and $\| \cdot \|_{\mathscr{H}}$ denote the corresponding inner product and norm in Hilbert space $\mathscr{H}$, respectively. In this chapter, I denotes at most countable set.

The most important notion for the convergence of a non-orthogonal sum over a general set of indices is unconditional convergence. A series is said to converge unconditionally if changing the order of terms in the sum does not affect the convergence of the series as it is stated in the next definition.

Definition 5.1.1 ([41])**.** *A series $\sum_{k\in I} f_k$, where $f_k \in \mathscr{H}$, $k \in I$, is said to be unconditionally convergent if the series $\sum_{k\in I} f_{\pi(k)}$ converges for all permutations π of I.*

In finite dimensional spaces, a series converges absolutely if and only if it converges unconditionally. However, in infinite dimensional spaces, absolute convergence implies unconditional convergence. For more details on unconditional convergence the reader can look at [41].

Definition 5.1.2 ([29, 41, 42])**.** *A family $\{f_k \in \mathscr{H} : k \in I\}$ is a basis for $\mathscr{H}$ if for every $f \in \mathscr{H}$ there are unique scalars α_k so that*

$$f = \sum_{k\in I} \alpha_k f_k. \tag{5.1.1}$$

A basis is unconditional if the series (5.1.1) *converges unconditionally. It is bounded if $0 < \inf_{k\in I} \|f_k\|_{\mathscr{H}} \leqslant \sup_{k\in I} \|f_k\|_{\mathscr{H}} < +\infty$. A basis is orthonormal if $\langle f_j, f_k \rangle_{\mathscr{H}} = \delta_{k,j}$, $k, j \in I$, where $\delta_{k,j}$ is Kronecker's*[1] *delta function*[2]*.*

Let us recall the statement about orthonormal bases. Therefore, let us introduce the following notations that will be used in the sequel. The set of all linear combinations of vectors $\{f_k : f_k \in \mathscr{H},\ k \in I\}$, i.e.

$$\left\{ \sum_{k\in I} \alpha_k f_k : \alpha_k \in \mathbb{C},\ f_k \in \mathscr{H},\ k \in I \right\}$$

is denoted by $\operatorname{span}\{f_k : f_k \in \mathscr{H},\ k \in I\}$, and its closure in $\mathscr{H}$ is denoted by $\overline{\operatorname{span}}\,\{f_k : f_k \in \mathscr{H},\ k \in I\}$.

Theorem 5.1.1 ([42])**.** *Let $\{f_k : f_k \in \mathscr{H},\ k \in I\}$ be an orthonormal basis in a Hilbert space $\mathscr{H}$. The following assertions are equivalent.*

(1) $\mathscr{H} = \overline{\operatorname{span}}\,\{f_k : f_k \in \mathscr{H},\ k \in I\}$.

(2) $f = \sum_{k\in I} \langle f, f_k \rangle_{\mathscr{H}} f_k$ *for all* $f \in \mathscr{H}$.

(3) $\|f\|^2_{\mathscr{H}} = \sum_{k\in I} |\langle f, f_k \rangle_{\mathscr{H}}|^2$ *for all* $f \in \mathscr{H}$ *(Parseval's*[3] *equality).*

(4) $\langle f, g \rangle_{\mathscr{H}} = \sum_{k\in I} \langle f, f_k \rangle_{\mathscr{H}} \overline{\langle g, f_k \rangle_{\mathscr{H}}}$ *for all* $f, g \in \mathscr{H}$.

Note, a Hilbert space has an orthonormal basis if and only if it is separable.

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[1] Leopold Kronecker (1823–1891) – German mathematician.

[2] $\delta_{k,j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$

[3] Michel Plancherel (1885–1967) – Swiss mathematician.

In order to introduce the definition of a Riesz basis, it is necessary first to introduce the definition of equivalent bases.

Definition 5.1.3 ([41])**.** *Basis* $\{f_k : k \in I\}$ *and* $\{g_k : k \in I\}$ *are equivalent for* $\mathscr{H}$ *if there exists a topological isomorphism* $S : \mathscr{H} \to \mathscr{H}$ *such that* $Sf_k = g_k$ *for every* $k \in I$*.*

Definition 5.1.4 ([41])**.** *A basis for* $\mathscr{H}$ *is called a Riesz basis if it is equivalent to some orthonormal basis for* $\mathscr{H}$*.*

Lemma 5.1.1 ([41])**.** *Let* $\{f_k : k \in I\}$ *be a Riesz basis for* $\mathscr{H}_1$ *and let* $S : \mathscr{H}_1 \to \mathscr{H}_2$ *be a topological isomorphism between two Hilbert spaces. Then,* $\{Sf_k : k \in I\}$ *is Riesz basis for* $\mathscr{H}_2$*.*

Proof. Assume that $\{f_k : k \in I\}$ is a Riesz basis for $\mathscr{H}_1$ and $S : \mathscr{H}_1 \to \mathscr{H}_2$ is a topological isomorphism between two Hilbert spaces. Then, since $\mathscr{H}_1$ has a basis, it is separable. Further, since $S : \mathscr{H}_1 \to \mathscr{H}_2$ is a topological isomorphism, it follows that $\mathscr{H}_2$ is also separable. Thus, there is an isometric isomorphism $T : \mathscr{H}_1 \to \mathscr{H}_2$. According to Definition 5.1.4, there is some orthonormal basis $\{g_k : k \in I\}$ for $\mathscr{H}_1$ and there is a topological isomorphism $K : \mathscr{H}_1 \to \mathscr{H}_1$ so that $Kg_k = f_k$ for every $k \in I$. Finally, $SKT^{-1} : \mathscr{H}_2 \to \mathscr{H}_2$ is a topological isomorphism and

$$SKT^{-1}(Tg_k) = SKg_k = Sf_k, \quad k \in I,$$

i.e. $\{Sf_k : k \in I\}$ is a Riesz basis for $\mathscr{H}_2$, since $\{Tg_k : k \in I\}$ is an othonormal basis for $\mathscr{H}_2$. □

A Bessel family is defined as follows.

Definition 5.1.5 ([41])**.** *A family* $\{f_k : f_k \in \mathscr{H}, k \in I\}$ *is said to be a Bessel family for* $\mathscr{H}$ *if*

$$\sum_{k \in I} |\langle f, f_k \rangle_{\mathscr{H}}|^2 < +\infty \quad \textit{for every } f \in \mathscr{H},$$

i.e. there is a constant $B > 0$ *so that* $\sum_{k \in I} |\langle f, f_k \rangle_{\mathscr{H}}|^2 \leqslant B\|f\|^2_{\mathscr{H}}$ *for every* $f \in \mathscr{H}$*.*

5.2 Frames in Hilbert spaces

Much effort has been invested in determining orthonormal bases that satisfy additional properties for various Hilbert spaces. It is often difficult to determine such an orthonormal basis, because the orthogonality condition is quite strong. As an alternative, this chapter presents the theory of frames. The advantage of frames over the orthonormal basis is that additional conditions can be more easily imposed. For more details about frame theory in Hilbert spaces, the reader can refer to [29, 33, 40, 41, 42, 44].

Definition 5.2.1 ([29, 40, 42]). *A family $\{f_k : k \in I\}$ of elements in a (separable) Hilbert space $\mathscr{H}$ is said to be a frame for $\mathscr{H}$ if there are positive constants A and B so that*

$$A\|f\|_{\mathscr{H}}^2 \leqslant \sum_{k\in I} |\langle f, f_k\rangle_{\mathscr{H}}|^2 \leqslant B\|f\|_{\mathscr{H}}^2 \quad \text{for every } f \in \mathscr{H}. \tag{5.2.1}$$

The constants A and B are called frame bounds. If the constants are equal, the frame is called tight. If the frame bounds are equal to 1, it is called a Parseval frame. A frame is exact if omitting one element in the family results it ceases to be a frame. A frame $\{f_k : k \in I\}$ in $\mathscr{H}$ is said to be fundamental if $\operatorname{span}\{f_k : k \in I\}$ *is dense in $\mathscr{H}$. The coefficients $\langle f, f_k\rangle_{\mathscr{H}}$, $k \in I$, are called the frame coefficients.*

Example 5.2.1. *Let $\{f_k : k \in I\}$ be an orthonormal basis for $\mathscr{H}$. Using Parseval's equality, it follows that it is a tight frame with frame bounds equal to 1, i.e. Parseval's frame. Moreover, it is an exact frame.*

Lemma 5.2.1 ([41]). *If $\{f_k : k \in I\}$ is a frame for $\mathscr{H}$, then $\{f_k : k \in I\}$ is complete in $\mathscr{H}$.*

Proof. Let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}$ and let $f \in \mathscr{H}$ so that $\langle f, f_k\rangle_{\mathscr{H}} = 0$ for every $k \in I$. Then, $A\|f\|_{\mathscr{H}}^2 \leqslant \sum_{k\in I} |\langle f, f_k\rangle_{\mathscr{H}}|^2 = 0$. □

Example 5.2.2. *Let $\{f_k : k \in I\}$ be an orthonormal basis for $\mathscr{H}$. Then, $\{f_1, \frac{f_2}{2}, \frac{f_3}{4}, \ldots\}$ is a complete orthonormal set. Moreover, it is a basis for $\mathscr{H}$, but it is not a frame (it does not have a lower frame bound).*

Lemma 5.2.2 ([40]). *Let $\{f_k : k \in I\}$ be a Parseval frame. If $\|f_k\|_{\mathscr{H}} = 1$ for every $k \in I$, then $\{f_k : k \in I\}$ is an orthonormal basis.*

Proof. Using the inequality (5.2.1), it follows that

$$1 = \|f_j\|_{\mathscr{H}}^2 = \sum_{k\in I} |\langle f_j, f_k\rangle_{\mathscr{H}}|^2 = \sum_{\substack{k\in I\\ k\neq j}} |\langle f_j, f_k\rangle_{\mathscr{H}}|^2 + 1, \quad j \in I.$$

Thus, $\langle f_j, f_k\rangle_{\mathscr{H}} = \delta_{k,j}$, $k, j \in I$. □

In order to better understand frames and reconstruction methods, some important operators should be studied, such as the analysis operator, the synthesis operator, and the frame operator.

Definition 5.2.2 ([29, 40]). *Let $\{f_k : k \in I\}$ be a subset of $\mathscr{H}$.*

(1) *The operator $C_o : \mathscr{H} \to \ell^2(I)$ defined by $C_o f = \{\langle f, f_k\rangle_{\mathscr{H}} : k \in I\}$, $f \in \mathscr{H}$, is called the coefficient operator (or the analysis operator).*

(2) *The operator $S_o : \ell^2(I) \to \mathscr{H}$ defined by $S_o\alpha = \sum_{k\in I} \alpha_k f_k$, $\alpha = (\alpha_k)_{k\in I} \in \ell^2(I)$, is called the synthesis operator (or the reconstruction operator).*

(3) *The operator $F_o : \mathscr{H} \to \mathscr{H}$ defined by $F_o f = \sum_{k\in I}\langle f, f_k\rangle_{\mathscr{H}} f_k$, $f \in \mathscr{H}$, is called the frame operator.*

Theorem 5.2.1 ([40, 41])**.** *Let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}$.*

(1) *The operator C_o is bounded with closed range.*

(2) *The operator S_o is the adjoint operator of C_o, i.e. $S_o = C_o^*$. As a consequence of this, S_o is bounded and holds $\|S_o\alpha\|_{\mathscr{H}} \leqslant \sqrt{B}\|\alpha\|_{\ell^2}$.*

(3) *The operator $F_o = C_o^* C_o = S_o S_o^*$ is a positive, invertible, self-adjoint operator and hold $A\mathbf{1}_{\mathscr{H}} \leqslant F_o \leqslant B\mathbf{1}_{\mathscr{H}}$ and $B^{-1}\mathbf{1}_{\mathscr{H}} \leqslant F_o^{-1} \leqslant A^{-1}\mathbf{1}_{\mathscr{H}}$. Specially, $\{f_k : k \in I\}$ is a tight frame if and only if $F_o = A\mathbf{1}_{\mathscr{H}}$.*

(4) *The frame bounds $A_{opt} = \|F_o^{-1}\|^{-1}$ and $B_{opt} = \|F_o\|$ are the optimal frame bounds.*

Proof. (1) Since $\{f_k : k \in I\}$ is a frame for $\mathscr{H}$, the inequality (5.2.1) implies the statement.

(2) Combining

$$\langle C_o^*\alpha, f\rangle_{\mathscr{H}} = \langle \alpha, C_o f\rangle_{\ell^2} = \sum_{k\in I}\alpha_k\overline{\langle f, f_k\rangle_{\mathscr{H}}} = \left\langle \sum_{k\in I}\alpha_k f_k, f\right\rangle_{\mathscr{H}} = \langle S_o\alpha, f\rangle_{\mathscr{H}},$$

with $\|C_o\| \leqslant \sqrt{B}$ (by (5.2.1)) leads to $S_o = C_o^*$ and $\|S_o\| \leqslant \sqrt{B}$.

(3) It is not difficult to see that $F_o = C_o^* C_o = S_o S_o^*$. Thus, the operator F_o is self-adjoint and positive. Using the inequality (5.2.1) and $\langle F_o f, f\rangle_{\mathscr{H}} = \sum_{k\in I}|\langle f, f_k\rangle_{\mathscr{H}}|^2$, it follows that $A\mathbf{1}_{\mathscr{H}} \leqslant F_o \leqslant B\mathbf{1}_{\mathscr{H}}$. Since A is a positive constant, F_o is invertible. Further, applying the operator F_o^{-1} to the previous inequalities gives $B^{-1}\mathbf{1}_{\mathscr{H}} \leqslant F_o^{-1} \leqslant A^{-1}\mathbf{1}_{\mathscr{H}}$, because F_o^{-1} is a positive operator and commutes with F_o.

(4) Since the norm of the positive operator F_o is given by

$$\|F_o\| = \sup\{\langle F_o f, f\rangle_{\mathscr{H}} : \|f\|_{\mathscr{H}} \leqslant 1\},$$

using the inequality (5.2.1), it follows that $B_{opt} = \|F_o\|$. A similar argument yields $A_{opt} = \|F_o^{-1}\|^{-1}$. □

Lemma 5.2.3 ([40])**.** *Let $\alpha = (\alpha_k)_{k\in I} \in \ell^2(I)$. If the set $\{f_k : k \in I\}$ is a frame for $\mathscr{H}$, then $\sum_{k\in I}\alpha_k f_k$ converges unconditionally to $f \in \mathscr{H}$.*

Proof. Let $\varepsilon > 0$. Choose $J \subseteq I$ so that $\sum_{k\notin J_0}|\alpha_k|^2 < \varepsilon B^{-1/2}$ for $J_0 \supseteq J$, and let $\alpha_{J_0} = \alpha \cdot \chi_{J_0} \in \ell^2(I)$, where χ_{J_0} is the characteristic function of J_0. Then, $\sum_{k\in J_0}\alpha_k f_k = S_o\alpha_{J_0}$ and

$$\left\| f - \sum_{k\in J_0}\alpha_k f_k\right\|_{\mathscr{H}} = \|S_o\alpha - S_o\alpha_{J_0}\|_{\mathscr{H}} = \|S_o(\alpha - \alpha_{J_0})\|_{\mathscr{H}} \leqslant \sqrt{B}\|\alpha - \alpha_{J_0}\|_{\ell^2} < \varepsilon,$$

by Theorem 5.2.1 (2). □

Lemma 5.2.4 ([41])**.** *Let $S : \mathscr{H}_1 \to \mathscr{H}_2$ be topological isomorphism, and let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}_1$. Then:*

(1) *$\{Sf_k : k \in I\}$ is a frame for $\mathscr{H}_2$, moreover, if A, B are the frame bounds for $\{f_k : k \in I\}$, then $A/\|S^{-1}\|^2$, $B\|S\|^2$ are the frame bounds for $\{Sf_k : k \in I\}$;*

(2) *if F_o is the frame operator for $\{f_k : k \in I\}$, then SF_oS^* is the frame operator for $\{Sf_k : k \in I\}$;*

(3) *$\{f_k : k \in I\}$ is exact if and only if $\{Sf_k : k \in I\}$ is an exact frame.*

Proof. Since

$$SF_oS^*g = S\Big(\sum_{k\in I}\langle S^*g, f_k\rangle_{\mathscr{H}_1} f_k\Big) = \sum_{k\in I}\langle g, Sf_k\rangle_{\mathscr{H}_2} Sf_k, \quad g \in \mathscr{H}_2,$$

(according to Theorem 5.2.1 (3)) the assertions (1) and (2) hold if

$$\frac{A}{\|S^{-1}\|^2}\mathbf{1}_{\mathscr{H}_2} \leqslant SF_oS^* \leqslant B\|S\|^2\mathbf{1}_{\mathscr{H}_2}. \tag{5.2.2}$$

Therefore, it suffices to prove that (5.2.2) holds.

Since F_o is the frame operator, it implies that $A\mathbf{1}_{\mathscr{H}_1} \leqslant F_o \leqslant B\mathbf{1}_{\mathscr{H}_1}$, by Theorem 5.2.1 (3). Thus,

$$A\|S^*g\|^2_{\mathscr{H}_1} \leqslant \langle SF_oS^*g, g\rangle_{\mathscr{H}_2} \leqslant B\|S^*g\|^2_{\mathscr{H}_1}, \quad g \in \mathscr{H}_2, \tag{5.2.3}$$

because $\langle SF_oS^*g, g\rangle_{\mathscr{H}_2} = \langle F_oS^*g, S^*g\rangle_{\mathscr{H}_1}$, $g \in \mathscr{H}_2$. On the other hand, $S : \mathscr{H}_1 \to \mathscr{H}_2$ is a topological isomorphism, it gives

$$\frac{\|g\|_{\mathscr{H}_2}}{\|S^{-1}\|} = \frac{\|g\|_{\mathscr{H}_2}}{\|S^{*^{-1}}\|} \leqslant \|S^*g\|_{\mathscr{H}_1} \leqslant \|S^*\|\|g\|_{\mathscr{H}_2} \leqslant \|S\|\|g\|_{\mathscr{H}_2}, \quad g \in \mathscr{H}_2. \tag{5.2.4}$$

Now, by (5.2.3) and (5.2.4), it follows that

$$\frac{A\|g\|^2_{\mathscr{H}_2}}{\|S^{-1}\|^2} \leqslant \langle SF_oS^*g, g\rangle_{\mathscr{H}_2} \leqslant B\|S\|^2\|g\|^2_{\mathscr{H}_2}, \quad g \in \mathscr{H}_2,$$

i.e. (5.2.2) holds.

Finally, since a topological isomorphism preserve complete and incomplete families, it follows that the assertion (3) holds. □

5.3 Dual frames and Riesz bases

A function f can be reconstructed using frame coefficients in the following way.

Lemma 5.3.1 ([40])**.** *Let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}$ with frame bounds A and B. Then, the set $\{F_o^{-1} f_k : k \in I\}$ is also a frame with frame bounds $\frac{1}{B}$ and $\frac{1}{A}$, and for every $f \in \mathscr{H}$ hold*

$$f = \sum_{k \in I} \langle f, F_o^{-1} f_k \rangle_{\mathscr{H}} f_k \tag{5.3.1}$$

and

$$f = \sum_{k \in I} \langle f, f_k \rangle_{\mathscr{H}} F_o^{-1} f_k. \tag{5.3.2}$$

Moreover, the both sums converge unconditionally.

Proof. Since

$$\begin{aligned}\sum_{k \in I} |\langle f, F_o^{-1} f_k \rangle_{\mathscr{H}}|^2 &= \sum_{k \in I} |\langle F_o^{-1} f, f_k \rangle_{\mathscr{H}}|^2 = \sum_{k \in I} \langle F_o^{-1} f, f_k \rangle_{\mathscr{H}} \overline{\langle F_o^{-1} f, f_k \rangle_{\mathscr{H}}} \\ &= \sum_{k \in I} \langle F_o^{-1} f, f_k \rangle_{\mathscr{H}} \langle f_k, F_o^{-1} f \rangle_{\mathscr{H}} = \langle F_o(F_o^{-1} f), F_o^{-1} f \rangle_{\mathscr{H}} = \langle F_o^{-1} f, f \rangle_{\mathscr{H}}\end{aligned}$$

and using Theorem 5.2.1 (3), it follows that

$$B^{-1} \|f\|_{\mathscr{H}}^2 \leqslant \langle F_o^{-1} f, f \rangle_{\mathscr{H}} = \sum_{k \in I} |\langle f, F_o^{-1} f_k \rangle_{\mathscr{H}}|^2 \leqslant A^{-1} \|f\|_{\mathscr{H}}^2.$$

Further,

$$f = F_o(F_o^{-1} f) = \sum_{k \in I} \langle F_o^{-1} f, f_k \rangle_{\mathscr{H}} f_k = \sum_{k \in I} \langle f, F_o^{-1} f_k \rangle_{\mathscr{H}} f_k$$

and

$$f = F_o^{-1}(F_o f) = F_o^{-1} \left(\sum_{k \in I} \langle f, f_k \rangle_{\mathscr{H}} f_k \right) = \sum_{k \in I} \langle f, f_k \rangle_{\mathscr{H}} F_o^{-1} f_k.$$

By Lemma 5.2.3, since $(\langle f, f_k \rangle_{\mathscr{H}})_{k \in I} \in \ell^2(I)$ and $(\langle f, F_o^{-1} f_k \rangle_{\mathscr{H}})_{k \in I} \in \ell^2(I)$, the both series converge unconditionally. □

Definition 5.3.1 ([41])**.** *A frame $\{F_o^{-1} f_k : k \in I\}$ is called the canonical dual frame of $\{f_k : k \in I\}$ for $\mathscr{H}$, where $\{f_k : k \in I\}$ is a frame for $\mathscr{H}$.*

Proposition 5.3.1 ([33, 42])**.** *Let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}$. If there are scalars $\beta_k \neq \langle f, F_o^{-1} f_k \rangle_{\mathscr{H}}$ so that $f = \sum_{k \in I} \beta_k f_k$, then*

$$\sum_{k \in I} |\beta_k|^2 = \sum_{k \in I} |\langle f, F_o^{-1} f_k \rangle_{\mathscr{H}}|^2 + \sum_{k \in I} |\langle f, F_o^{-1} f_k \rangle_{\mathscr{H}} - \beta_k|^2.$$

Proof. Denote $\alpha_k = \langle f, F_o^{-1} f_k \rangle_{\mathscr{H}}$, $k \in I$. Note that $\langle f_k, F_o^{-1} f \rangle_{\mathscr{H}} = \langle F_o^{-1} f_k, f \rangle_{\mathscr{H}} = \overline{\alpha_k}$, $k \in I$. Since $f = \sum_{k \in I} \alpha_k f_k$, it implies that

$$\langle f, F_o^{-1} f \rangle_{\mathscr{H}} = \left\langle \sum_{k \in I} \alpha_k f_k, F_o^{-1} f \right\rangle_{\mathscr{H}} = \sum_{k \in I} |\alpha_k|^2. \tag{5.3.3}$$

On the other hand, since $f = \sum_{k\in I} \beta_k f_k$, it gives

$$\langle f, F_o^{-1} f\rangle_{\mathscr{H}} = \Big\langle \sum_{k\in I} \beta_k f_k, F_o^{-1} f \Big\rangle_{\mathscr{H}} = \sum_{k\in I} \beta_k \overline{\alpha_k}. \tag{5.3.4}$$

Now, from (5.3.3) and (5.3.4), it follows that $\sum_{k\in I} |\alpha_k|^2 = \sum_{k\in I} \beta_k \overline{\alpha_k}$. Therefore,

$$\begin{aligned}\sum_{k\in I} |\alpha_k|^2 + \sum_{k\in I} |\alpha_k - \beta_k|^2 &= \sum_{k\in I} |\alpha_k|^2 + \sum_{k\in I} \left(|\alpha_k|^2 - \alpha_k \overline{\beta_k} - \overline{\alpha_k}\beta_k + |\beta_k|^2\right) \\ &= \sum_{k\in I} |\beta_k|^2.\end{aligned}$$

Hence, the statement holds. □

Remark 5.3.1. *According to Proposition* 5.3.1, *the coefficients* $\langle f, F_o^{-1} f_k\rangle_{\mathscr{H}}$, $k \in I$, *in the equality* (5.3.1) *are not unique in the general case.*

Theorem 5.3.1 ([33, 41, 42])**.** *Let* $\{f_k : k \in I\}$ *be a frame for* $\mathscr{H}$*.*

(1) *If* $\langle f_j, F_o^{-1} f_j\rangle_{\mathscr{H}} \neq 1$ *for some* $j \in I$, *then* $\{f_k : k \in I,\ k \neq j\}$ *is a frame.*

(2) *If* $\langle f_j, F_o^{-1} f_j\rangle_{\mathscr{H}} = 1$ *for some* $j \in I$, *then* $\{f_k : k \in I,\ k \neq j\}$ *is incomplete.*

Proof. Let $j \in I$ be fixed and denote $\alpha_k = \langle f_j, F_o^{-1} f_k\rangle_{\mathscr{H}} = \langle F_o^{-1} f_j, f_k\rangle_{\mathscr{H}}$, $k \in I$. Now, $f_j = \sum_{k\in I} \alpha_k f_k$ and $f_j = \sum_{k\in I} \beta_k f_k$, where $\beta_k = \delta_{k,j}$, $k \in I$. Thus, by Proposition 5.3.1,

$$1 = \sum_{k\in I} |\beta_k|^2 = \sum_{k\in I} |\alpha_k|^2 + \sum_{k\in I} |\alpha_k - \beta_k|^2 = \sum_{\substack{k\in I\\ k\neq j}} |\alpha_k|^2 + |\alpha_j|^2 + \sum_{\substack{k\in I\\ k\neq j}} |\alpha_k|^2 + |\alpha_j - 1|^2. \tag{5.3.5}$$

Assume that $\alpha_j = 1$. Then, $\sum_{k\in I,\, k\neq j} |\alpha_k|^2 = 0$ and thus $\alpha_k = \langle F_o^{-1} f_j, f_k\rangle_{\mathscr{H}} = 0$, $k \neq j$. Since $\langle F_o^{-1} f_j, f_k\rangle_{\mathscr{H}} = \alpha_j = 1$, it implies that $F_o^{-1} f_j \neq \mathbf{0}$. Therefore, $\{f_k : k \in I,\ k \neq j\}$ is incomplete.

If $\alpha_j \neq 1$, then $f_j = \frac{1}{1-\alpha_j} \sum_{k\in I,\, k\neq j} \alpha_k f_k$. Thus,

$$|\langle f, f_j\rangle_{\mathscr{H}}|^2 = \left| \frac{1}{1-\alpha_j} \sum_{\substack{k\in I\\ k\neq j}} \alpha_k \langle f, f_k\rangle_{\mathscr{H}} \right|^2 \leqslant C \sum_{\substack{k\in I\\ k\neq j}} \langle f, f_k\rangle_{\mathscr{H}}, \quad f \in \mathscr{H},$$

where $C = |1-\alpha_j|^{-2} \sum_{k\in I,\, k\neq j} |\alpha_k|^2$. Now, since

$$\sum_{k\in I} |\langle f, f_k\rangle_{\mathscr{H}}|^2 = \sum_{\substack{k\in I\\ k\neq j}} |\langle f, f_k\rangle_{\mathscr{H}}|^2 + |\langle f, f_j\rangle_{\mathscr{H}}|^2 \leqslant (1+C) \sum_{\substack{k\in I\\ k\neq j}} |\langle f, f_k\rangle_{\mathscr{H}}|^2,$$

it implies that $\{f_k : k \in I,\ k \neq j\}$ is a frame with frame bounds $\frac{A}{1+C}, B$. □

Corollary 5.3.1 ([41, 42])**.** *Let* $\{f_k : k \in I\}$ *be an exact frame for* $\mathscr{H}$*. Then,* $\{f_k : k \in I\}$ *and* $\{F_o^{-1} f_k : k \in I\}$ *are biorthonormal, i.e.* $\langle f_j, F_o^{-1} f_k\rangle_{\mathscr{H}} = \delta_{k,j}$.

Proof. Let $\{f_k : k \in I\}$ be an exact frame for $\mathscr{H}$. Then, according to Theorem 5.3.1, $\langle f_j, F_o^{-1} f_j\rangle_{\mathscr{H}} = 1$, $j \in I$. Therefore, by the equality (5.3.5) (with $\alpha_k = \langle f_j, F_o^{-1} f_k\rangle_{\mathscr{H}}$ and $\beta_k = \langle f_j, F_o^{-1} f_j\rangle_{\mathscr{H}}$), it follows that $\langle f_j, F_o^{-1} f_k\rangle_{\mathscr{H}} = \langle F_o^{-1} f_j, f_k\rangle_{\mathscr{H}} = 0$ for $k, j \in I$, $k \neq j$. □

The conditions for the uniqueness of the sequence of coefficients $\alpha \in \ell^2(I)$ in (5.3.1) are derived in the following theorem.

Let G denote the Gram's matrix with elements $G_{k,j} = \langle f_j, f_k \rangle_{\mathscr{H}}$, $k, j \in I$.

Theorem 5.3.2 ([40]). *Let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}$. The following conditions are equivalent.*

(1) *The sequence of coefficients $\alpha \in \ell^2(I)$ in the equality* (5.3.1) *is unique.*

(2) *The operator C_o maps $\mathscr{H}$ onto $\ell^2(I)$.*

(3) *There are positive constants A' and B' so that*

$$A' \|\alpha\|_{\ell^2} \leqslant \Big\| \sum_{k \in I} \alpha_k f_k \Big\|_{\mathscr{H}} \leqslant B' \|\alpha\|_{\ell^2} \tag{5.3.6}$$

for every finite sequence $\alpha = (\alpha_k)_{k \in I}$.

(4) *The image of an orthonormal basis $\{g_k : k \in I\}$ under the bounded invertable operator $Q : \mathscr{H} \to \mathscr{H}$ is the frame $\{f_k : k \in I\}$.*

(5) *The Gram's matrix defines on $\ell^2(I)$ a positive invertabile operator.*

Proof. Note, since $\{f_k : k \in I\}$ is a frame, the operator C_o is one-to-one with the closed range (by Theorem 5.2.1) and the operator F_o is onto (by (5.3.1)).

The sequence of coefficients $\alpha \in \ell^2(I)$ in the equality (5.3.1) is unique if and only if the operator S_o is one-to-one if and only if its adjoint $S_o^* = C_o$ is onto. Hence, (1) is equivalent to (2).

(1) $\Rightarrow$ (3) According to Theorem 5.2.1 (2), it follows that the constant B' in the inequality (5.3.6) exists. The existence of A' follows from the fact that the operator S_o^{-1} is continuous (the operator S_o^{-1} is continuous by the Open Mapping Theorem, because S_o is a bijective operator).

(3) $\Rightarrow$ (4) Let $\{g_k : k \in I\}$ be an orthonormal basis for $\mathscr{H}$ and let $Qf = \sum_{k \in I} \alpha_k f_k$, where $f = \sum_{k \in I} \alpha_k g_k$. Then, $\|f\|_{\mathscr{H}} = \|\alpha\|_{\ell^2}$ and

$$A' \|f\|_{\mathscr{H}} = A' \|\alpha\|_{\ell^2} \leqslant \Big\| \sum_{k \in I} \alpha_k f_k \Big\|_{\mathscr{H}} = \|Qf\|_{\mathscr{H}} \leqslant B' \|\alpha\|_{\ell^2} = B' \|f\|_{\mathscr{H}},$$

for every $f \in \mathscr{H}$. Therefore, Q is well defined invertible operator and for every $k \in I$, $Qg_k = f_k$.

(4) $\Rightarrow$ (1) Let $\{g_k : k \in I\}$ be an orthonormal basis for $\mathscr{H}$, and let Q be a bounded invertable operator such that $Qg_k = f_k$, $k \in I$. Then,

$$\sum_{k \in I} \alpha_k f_k = Q\Big(\sum_{k \in I} \alpha_k g_k \Big) = \mathbf{0} \quad \Leftrightarrow \quad \sum_{k \in I} \alpha_k g_k = \mathbf{0} \quad \Leftrightarrow \quad \alpha_k = 0,\ k \in I.$$

Hence, the sequence of coefficients $\alpha \in \ell^2(I)$ in the equality (5.3.1) is unique.

(3) $\Leftrightarrow$ (5) Let $\alpha = (\alpha_k)_{k \in I}$ be an arbitrary sequence. Then,

$$\langle G\alpha, \alpha \rangle_{\ell^2} = \sum_{k,j \in I} \langle f_k, f_j \rangle_{\mathscr{H}} \alpha_k \overline{\alpha_j} = \Big\| \sum_{k \in I} \alpha_k f_k \Big\|_{\mathscr{H}}^2 .$$

Therefore, by the equality (5.3.6), the operator G is positive and invertable on $\ell^2(I)$. □

Now, Definition 5.1.4 can be reformulated as follows (Definition 5.3.2 (2)).

Definition 5.3.2 ([40])**.** (1) *The set* $\{f_k : k \in I\}$ *is said to be a Riesz family for* $\mathscr{H}$ *if it satisfies the condition* (3) *of Theorem* 5.3.2.

(2) *The set* $\{f_k : k \in I\}$ *is said to be a Riesz basis for* $\mathscr{H}$ *if* $\{f_k : k \in I\}$ *is a frame for* $\mathscr{H}$ *and satisfies the conditions of Theorem* 5.3.2.

Hence, a Riesz basis is a Riesz family which is complete in $\mathscr{H}$.

Theorem 5.3.3 ([76])**.** *A family* $\{f_k : k \in I\}$ *is a Riesz basis for* $\mathscr{H}$ *if and only if* $\{f_k : k \in I\}$ *is a bounded unconditional basis for* $\mathscr{H}$.

Therefore, a bounded unconditional basis is equivalent to an orthonormal basis.

Theorem 5.3.4 ([29])**.** *Let* $\{f_k : k \in I\}$ *be a frame for* $\mathscr{H}$*. Then, the following assertions are equivalent.*

(1) $\{f_k : k \in I\}$ *is Riesz basis for* $\mathscr{H}$.

(2) $\{f_k : k \in I\}$ *is an exact frame for* $\mathscr{H}$.

(3) $\{f_k : k \in I\}$ *is a basis for* $\mathscr{H}$.

(4) *If* $\sum_{k\in I} \alpha_k f_k = \mathbf{0}$ *for some* $(\alpha_k)_{k\in I} \in \ell^2(I)$*, then* $\alpha_k = 0$*,* $k \in \mathbb{N}$.

Recall that a positive linear and continuous operator has the positive and continuous square root. Therefore, F_o^{-1} has the positive and continuous square root $F_o^{-1/2} = (F_o^{-1})^{1/2}$.

Lemma 5.3.2 ([28, 40])**.** *Let* $\{f_k : k \in I\}$ *be a frame for* $\mathscr{H}$*. Then,*

(1) *the set* $\{F_o^{-1/2} f_k : k \in I\}$ *is a Parseval frame;*

(2) *the inverse frame operator* F_o^{-1} *is given by*

$$F_o^{-1} f = \sum_{k\in I} \langle f, F_o^{-1} f_k\rangle_{\mathscr{H}} F_o^{-1} f_k.$$

Proof. (1) Since the frame operator F_o is positive, it follows that $F_o^{-1/2}$ is well defined and a positive operator. Further, since

$$f = F_o^{-1/2} F_o(F_o^{-1/2} f) = \sum_{k\in I} \langle f, F_o^{-1/2} f_k\rangle_{\mathscr{H}} F_o^{-1/2} f_k,$$

it implies that $\|f\|^2_{\mathscr{H}} = \langle f, f\rangle_{\mathscr{H}} = \sum_{k\in I} \left|\langle f, F_o^{-1/2} f_k\rangle_{\mathscr{H}}\right|^2$. Hence, $\{F_o^{-1/2} f_k : k \in I\}$ is a Parseval frame.

(2) Obviously

$$F_o^{-1} f = F_o^{-1} F_o(F_o^{-1} f) = \sum_{k\in I} \langle f, F_o^{-1} f_k\rangle_{\mathscr{H}} F_o^{-1} f_k,$$

which had to be proved. □

Definition 5.3.3 ([29])**.** *A frame for* $\mathscr{H}$ *is called overcomplete frame if it is not a basis for* $\mathscr{H}$.

Definition 5.3.4 ([29]). *Let $\{f_k : k \in I\}$ be a frame for $\mathscr{H}$. A family $\{g_k : k \in I\} \subset \mathscr{H}$ such that*

$$f = \sum_{k \in I} \langle f, g_k \rangle_{\mathscr{H}} f_k, \quad f \in \mathscr{H}, \tag{5.3.7}$$

is called a dual frame of $\{f_k : k \in I\}$.

Theorem 5.3.5 ([29]). *A frame $\{f_k : k \in I\}$ for $\mathscr{H}$ has a unique dual frame if and only if it is exact.*

Proof. The necessary condition is proved by Proposition 5.3.1 and Corollary 5.3.1. In the opposite direction, suppose, contrary to our claim, that $\{f_k : k \in I\}$ is an inexact frame. The proof consists of two cases. First, assume that $f_j = \mathbf{0}$ for some $j \in I$. Then, $F_o^{-1} f_j = \mathbf{0}$. Let $g_k = F_o^{-1} f_k$, $k \in I$, $k \neq j$, and let $g_j \neq \mathbf{0}$ be arbitrary chosen. Thus, the equation (5.3.1) holds and $\{g_k : k \in I\} \neq \{F_o^{-1} f_k : k \in I\}$. In this case, a new dual frame is obtained (it is not the canonical dual), a contradiction.

Now, assume that $f_k \neq \mathbf{0}$ for every $k \in \mathbb{N}$. Then, since $\{f_k : k \in I\}$ is overcomplete, there is a sequences $(\beta_k)_{k \in I} \in \ell^2(I) \setminus \{\mathbf{0}\}$ so that

$$\sum_{k \in I} \beta_k f_k = \mathbf{0}, \tag{5.3.8}$$

by Theorem 5.3.4. Thus, there exists $j \in I$ such that $\beta_j \neq 0$. Therefore, from (5.3.8), it follows that

$$f_j = \sum_{\substack{k \in I \\ k \neq j}} -\frac{\beta_k}{\beta_j} f_k.$$

Hence, $\{f_k : k \in I, \, k \neq j\}$ is complete in $\mathscr{H}$ and thus it is a frame, by Theorem 5.3.1. Further, let $\{g_k : k \in I, \, k \neq j\}$ be its canonical dual frame and set $g_j = \mathbf{0}$. Then, $\{g_k : k \in I\}$ is a dual frame for $\{f_k : k \in I\}$, but it is not the canonical dual since $g_j = \mathbf{0}$ while $F_o^{-1} f_j \neq \mathbf{0}$, which is impossible. □

Some statements about dual frames are given in the following lemmas.

Lemma 5.3.3 ([29]). *Let $\{f_k : k \in I\}$ and $\{g_k : k \in I\}$ be Bessel famillies for $\mathscr{H}$. Then, the following assertions are equivalent.*

(1) $f = \sum_{k \in I} \langle f, g_k \rangle_{\mathscr{H}} f_k$, $f \in \mathscr{H}$.

(2) $f = \sum_{k \in I} \langle f, f_k \rangle_{\mathscr{H}} g_k$, $f \in \mathscr{H}$.

(3) $\langle f, g \rangle_{\mathscr{H}} = \sum_{k \in I} \langle f, f_k \rangle_{\mathscr{H}} \langle g_k, g \rangle_{\mathscr{H}}$, $f, g \in \mathscr{H}$.

Lemma 5.3.4 ([29]). *Let $\{f_k : k \in I\}$ and $\{g_k : k \in I\}$ be two Bessel families. If*

$$\|f\|_{\mathscr{H}}^2 = \sum_{k \in I} \langle f, f_k \rangle_{\mathscr{H}} \langle g_k, f \rangle_{\mathscr{H}}$$

holds for every f from a dense subspace of $\mathscr{H}$, then $\{f_k : k \in I\}$ and $\{g_k : k \in I\}$ are dual frames.

Lemma 5.3.5 ([29]). *Let $\{f_k : k \in I\}$ and $\{g_k : k \in I\}$ be dual frames for $\mathscr{H}$. If $S : \mathscr{H} \to \mathscr{H}$ is a unitary operator, then $\{S f_k : k \in I\}$ and $\{S g_k : k \in I\}$ are also dual frames for $\mathscr{H}$.*

Chapter 6

Shift-invariant subspaces of Sobolev spaces

After a short trip through the theoretical basis for the dissertation, this chapter presents the research results of this doctoral dissertation. These are the results of papers [6]–[8]. The initial results of this dissertation derive from Bownik's approach [23] which is now applied to the Sobolev spaces H^r, $r \in \mathbb{R}$, leading from the Sobolev spaces H^r to the weighted sequence spaces ℓ_r^2. In this chapter, unless otherwise stated, the statements are valid for every $r \in \mathbb{R}$.

6.1 Notations and basic assertions

Let $\mathscr{A}_r \subset H^r$, i.e. $\mathscr{A}_r = \{f \in \mathscr{S}' : \widehat{f} = \widehat{g}\mu_{-r} \text{ for some } g \in \mathscr{A}\}$ and $E_r(\mathscr{A}_r) = \{T_q f : f \in \mathscr{A}_r,\, q \in \mathbb{Z}^n\}$, where $\mathscr{A} \subset L^2$ is at most countable set. Obviously, $E_r(\mathscr{A}_r) \subset H^r$. Note, in the continuation I denotes a finite set or $I = \mathbb{N}$. Therefore, the notations $\mathscr{A}_{r,I} = \{f_k \in \mathscr{S}' : \widehat{f_k} = \widehat{g_k}\mu_{-r} \text{ for some } g_k \in \mathscr{A}_I,\, k \in I\}$ will also be used when an index set I is given. If $I = \{1, \dots, m\}$, then the notations $\mathscr{A}_{r,m}$ will be used. The SI subspace of Sobolev space will be denoted by $V_r \subset H^r$, $r \in \mathbb{R}$ (see Definition 1.0.1).

Further, let $S_r(\mathscr{A}_r) = \overline{\operatorname{span}}\, E_r(\mathscr{A}_r) = \overline{\operatorname{span}}\left\{(\mathbf{1} - \frac{1}{4\pi^2}\Delta)^{-r/2} T_q g : g \in \mathscr{A},\, q \in \mathbb{Z}^n\right\}$. It is not difficult to see that the space $S_r(\mathscr{A}_r)$ generated by $\mathscr{A}_r$ is a SI space.

Definition 6.1.1 ([6, 23])**.** *A SI space V_r is called a finitely generated shift-invariant (FSI) space if V_r is generated by a finite set of functions, i.e. $V_r = S_r(\mathscr{A}_{r,m})$. A SI space V_r is called a principal shift-invariant (PSI) space if V_r is generated by only one function, i.e. $V_r = S_r(f) = S_r(\{f\})$.*

Similarly, like in Bownik [23], a new space and a new mapping are introduced.

Definition 6.1.2 ([6])**.** *The space of all vector valued measurable functions $H : \mathbb{T}^n \to \ell_r^2$ such that*

$$\int_{\mathbb{T}^n} \|H(t)\|_{\ell_r^2}^2 \,\mathrm{d}t < +\infty$$

is denoted by $\mathscr{H}^r(\mathbb{T}^n, \ell_r^2)$, or shorter $\mathscr{H}^r$.

Lemma 6.1.1 ([6]). *The space $\mathscr{H}^r$ is a Hilbert space with the inner product*

$$\langle H_1, H_2\rangle_{\mathscr{H}^r} = \int_{\mathbb{T}^n} \langle H_1(t), H_2(t)\rangle^2_{\ell^2_r} \,\mathrm{d}t,$$

and the corresponding norm

$$\|H\|_{\mathscr{H}^r} = \left(\int_{\mathbb{T}^n} \|H(t)\|^2_{\ell^2_r} \,\mathrm{d}t\right)^{1/2}.$$

Lemma 6.1.2 ([6]). *The mapping $\mathscr{T}_r : H^r \to \mathscr{H}^r$ defined by*

$$\mathscr{T}_r f(t) = \left(\frac{\widehat{g}(t+q)}{\mu_r(q)}\right)_{q\in\mathbb{Z}^n}, \quad t \in \mathbb{T}^n,\ f \in H^r,$$

where $\left(\mathbf{1} - \frac{1}{4\pi^2}\Delta\right)^{r/2} f = g \in L^2$, is an isometric isomorphism. Moreover, for every $f \in \mathscr{S}$ holds $\mathscr{T}_r T_q f(\cdot) = M_{-q}\mathscr{T}_r f(\cdot)$, $q \in \mathbb{Z}^n$.

Proof. Theorem 3.5.2 brings enough to prove the statement for an arbitrary function $f \in \mathscr{S}$. Therefore, let $f \in \mathscr{S}$ and let $\widehat{f} = \widehat{g}\mu_{-r}$. Then,

$$\begin{aligned}\|\mathscr{T}_r f\|^2_{\mathscr{H}^r} &= \int_{\mathbb{T}^n} \|\mathscr{T}_r f(t)\|^2_{\ell^2_r} \,\mathrm{d}t = \int_{\mathbb{T}^n} \left\|\left(\frac{\widehat{g}(t+q)}{\mu_r(q)}\right)_{q\in\mathbb{Z}^n}\right\|^2_{\ell^2_r} \mathrm{d}t = \int_{\mathbb{T}^n} \sum_{q\in\mathbb{Z}^n} |\widehat{g}(t+q)|^2 \,\mathrm{d}t \\ &= \int_{\mathbb{R}^n} |\widehat{g}(t)|^2 \,\mathrm{d}t = \int_{\mathbb{R}^n} |\widehat{f}(t)|^2 \mu_r^2(t) \,\mathrm{d}t = \|f\|^2_{H^r},\end{aligned}$$

where Theorem 2.3.4 and Lemma 4.4.1 are used. The second part of the assertion follows by the theorems 3.2.2 and 2.3.4. □

Note, if $r = 0$, then $\mathscr{T}_0 f(t) = \left(\widehat{f}(t+q)\right)_{q\in\mathbb{Z}^n} = \mathscr{T} f(t)$, $t \in \mathbb{T}^n$, $f \in H^0 = L^2$, i.e. $\mathscr{H}^0 = L^2(\mathbb{T}^n, \ell^2)$. The next assertion is obvious.

Lemma 6.1.3 ([6]). *The diagram of isometries*

$$\begin{array}{ccc} L^2 & \overset{\mathscr{T}}{\longrightarrow} & \mathscr{H}^0 \\ \downarrow \alpha_r & & \downarrow \beta_r \\ H^r & \overset{\mathscr{T}_r}{\longrightarrow} & \mathscr{H}^r \end{array}$$

commutes, where $\alpha_r(g) = \mathscr{F}^{-1}\left(\frac{\widehat{g}(\cdot)}{\mu_r(\cdot)}\right)$ and $\beta_r\left(\left(\widehat{g}(\cdot+q)\right)_{q\in\mathbb{Z}^n}\right) = \left(\frac{\widehat{g}(\cdot+q)}{\mu_r(q)}\right)_{q\in\mathbb{Z}^n}$.

The following definition is analogous to Definition 1.0.3.

Definition 6.1.3 ([6]). *A mapping*

$$J_r : \mathbb{T}^n \to \left\{\textit{closed subspaces of } \ell^2_r\right\}$$

($t \mapsto J_r(t)$, $t \in \mathbb{T}^n$) is called the range function.

The range function J_r is measurable if for any $a, b \in \ell^2_r$, $t \mapsto \langle P_{J_r(t)}(a), b\rangle_{\ell^2_r}$ is a measurable scalar function (i.e. if $P_{J_r(t)}$, $t \in \mathbb{T}^n$, are weakly operator measurable), where $P_{J_r(t)} : \ell^2_r \to J_r(t)$, $t \in \mathbb{T}^n$, are the associated orthogonal projections. Note, in the separable Hilbert space weak and strong measurability are equivalent.

In the continuation, unless stated otherwise, the family $\{e_\nu : \nu \in \mathbb{Z}^n\}$ denotes the standard basis for ℓ_r^2 (the standard form of representation of the vector e_ν is $(0,\dots,0,1,0,\dots,0)$, where 1 is on the ν-th position). Define the subspace of $\mathscr{H}^r$ by

$$N_{J_r} = \big\{H \in \mathscr{H}^r \,:\, H(t) \in J_r(t) \text{ for a.e. } t \in \mathbb{T}^n\big\}.$$

Lemma 6.1.4 ([6]). *Assume that J_r is a measurable range function. Let*

$$P_{J_r} : \mathbb{T}^n \to \big\{\textit{space of projections of } \ell_r^2 \textit{ onto closed subspaces of } \ell_r^2\big\},$$

so that $P_{J_r(t)} : \ell_r^2 \to J_r(t)$ for a.e. $t \in \mathbb{T}^n$ be the associated orthogonal projections, and let P_r be the orthogonal projection

$$\mathscr{H}^r \ni H \mapsto P_r(H) \in N_{J_r}$$

such that $\big(P_rH\big)(t) \in J_r(t)$ for a.e. $t \in \mathbb{T}^n$. Then, for every $H \in \mathscr{H}^r$ holds

$$\big(P_rH\big)(t) = P_{J_r(t)}\big(H(t)\big) \quad \textit{for a.e. } t \in \mathbb{T}^n. \tag{6.1.1}$$

Proof. Let $P'_r : \mathscr{H}^r \to \mathscr{H}^r$ be given by

$$\big(P'_rH\big)(t) = P_{J_r(t)}\big(H(t)\big) \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Since $\|P_{J_r(t)}\|_{\ell_r^2} \leqslant 1$, the measurable vector function $P_{J_r(t)}\big(H(t)\big)$ belongs to $\mathscr{H}^r$. It is clear that P'_r is an orthogonal projection with range N'_r. To prove $N_{J_r} = N'_r$, it only remains to verify $N_{J_r} \subseteq N'_r$. Suppose, contrary to our claim, that there exists $\mathbf{0} \neq H_1 \in N_{J_r}$ orthogonal to N'_r. Then,

$$\begin{aligned} 0 &= \int_{\mathbb{T}^n} \big\langle \big(P'_rH\big)(t), H_1(t)\big\rangle_{\ell_r^2} \,\mathrm{d}t = \int_{\mathbb{T}^n} \big\langle P_{J_r(t)}\big(H(t)\big), H_1(t)\big\rangle_{\ell_r^2} \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \big\langle H(t), P_{J_r(t)}\big(H_1(t)\big)\big\rangle_{\ell_r^2} \,\mathrm{d}t \quad \text{for all } H \in \mathscr{H}^r. \end{aligned}$$

Since $H_1(t) \in J_r(t)$, it follows that $H_1(t) = P_{J_r(t)}\big(H_1(t)\big) = \mathbf{0}$ for a.e. $t \in \mathbb{T}^n$, which is impossible. $\square$

Lemma 6.1.5 ([6]). *Let J_r be an arbitrary range function (not necessarily measurable). Then, the space N_{J_r} is a closed subspace of $\mathscr{H}^r$. Moreover, if for some measurable range functions J_r and K_r holds $N_{J_r} = N_{K_r}$, then $J_r(t) = K_r(t)$ for a.e. $t \in \mathbb{T}^n$.*

Proof. Let $(H_\nu)_{\nu\in\mathbb{N}} \subset N_{J_r}$ be a sequence such that $\lim_{\nu\to+\infty} H_\nu = H$ in $\mathscr{H}^r$. Then, there exists a subsequence such that $\lim_{j\to+\infty} H_{\nu_j}(t) = H(t)$ in ℓ_r^2 for a.e. $t \in \mathbb{T}^n$. Thus, using the fact that $J_r(t)$ is closed, it follows that $H \in N_{J_r}$.

Further, let for some measurable range functions J_r and K_r holds $N_{J_r} = N_{K_r}$ and let P_{J_r} and P_{K_r} be associated orthogonal projections, respectively. Let $H(t) = e_q$ for some $q \in \mathbb{Z}^n$ be a constant function, where $e_q \in \ell_r^2$ is the standard vector. Applying Lemma 6.1.4 to $H(t) = e_q$ gives $P_{J_r(t)}(e_q) = \big(P_re_q\big)(t) = P_{K_r(t)}(e_q)$, i.e. $P_{J_r(t)}(e_q) = P_{K_r(t)}(e_q)$ for a.e. $t \in \mathbb{T}^n$, $q \in \mathbb{Z}^n$. Hence, $P_{J_r(t)} = P_{K_r(r)}$ for a.e. $t \in \mathbb{T}^n$. $\square$

Remark 6.1.1. *The equality* (6.1.1) *is equivalent to $\mathscr{T}_r\big(P_{V_r}f\big)(t) = P_{J_r(t)}\big(\mathscr{T}_rf(t)\big)$ for a.e. $t \in \mathbb{T}^n$, $f \in H^r$, where P_{V_r} is othogonal projection on V_r (see* [5]*).*

The following theorem is extremely important, SI spaces are connected with the range function and vice versa. Range functions are said to be equal if they are equal almost everywhere.

Theorem 6.1.1 ([6])**.** *A space $V_r \subset H^r$ is SI if and only if there is a measurable range function J_r so that*

$$V_r = \big\{ f \in H^r \,:\, \mathscr{T}_r f(t) \in J_r(t) \text{ for a.e. } t \in \mathbb{T}^n \big\}. \tag{6.1.2}$$

The relationship between V_r and J_r is one-to-one. If $V_r = S_r(\mathscr{A}_{r,I})$, where $\mathscr{A}_{r,I} \subset H^r$, then

$$J_r(t) = \overline{\operatorname{span}}\,\{\mathscr{T}_r f(t) : f \in \mathscr{A}_{r,I}\}. \tag{6.1.3}$$

Proof. Suppose $V_r = S_r(\mathscr{A}_{r,I})$, where $\mathscr{A}_{r,I} \subset H^r$, is a SI space and $J_r(t)$ is given by (6.1.3). Let $N_r = \mathscr{T}_r V_r$. According to Lemma 6.1.2, a subspace $V_r \subset H^r$ is SI if and only if N_r is a closed subspace of $\mathscr{H}^r$ (closed under multiplication by exponentials). Thus, for every $H \in N_r$ there exists a sequence $(H_\nu)_{\nu \in \mathbb{N}}$ such that

$$\mathscr{T}_r^{-1} H_\nu \in \operatorname{span}\{T_q f : f \in \mathscr{A}_{r,I},\ q \in \mathbb{Z}^n\} \quad \text{and} \quad \lim_{\nu \to +\infty} H_\nu = H.$$

From Lemma 6.1.2, it follows that $H_\nu(t) \in J_r(t)$ for every $\nu \in \mathbb{N}$. Hence, $H(t) \in J_r(t)$ and finally $N_r \subseteq N_{J_r}$.

Suppose that there exists $\mathbf{0} \neq H_1 \in \mathscr{H}^r$ orthogonal to N_r. Since for all $H \in \mathscr{T}_r \mathscr{A}_{r,I}$ and $q \in \mathbb{Z}^n$ hold

$$\int_{\mathbb{T}^n} \mathrm{e}^{-2\pi i \langle t,q \rangle} \langle H(t), H_1(t) \rangle_{\ell^2_r} \,\mathrm{d}t = \int_{\mathbb{T}^n} \langle \mathrm{e}^{-2\pi i \langle t,q \rangle}\, H(t), H_1(t) \rangle_{\ell^2_r} \,\mathrm{d}t = 0,$$

it follows that $\langle H(t), H_1(t) \rangle_{\ell^2_r} = 0$ for a.e. $t \in \mathbb{T}^n$ and all $H \in \mathscr{T}_r \mathscr{A}_{r,I}$. Thus, $H_1(t) \in (J_r(t))^\perp$ for a.e. $t \in \mathbb{T}^n$ and therefore there does not exist $\mathbf{0} \neq H_1 \in N_{J_r}$ orthogonal to N_r. This clearly forces $N_r = N_{J_r}$.

It remains the measurability of J_r (given by (6.1.3)) to be proved. From what has already been proved, for $H \in \mathscr{H}^r$ and the orthogonal projection P_r of $\mathscr{H}^r$ onto N_r, it follows that $H(t) - \big(P_r H\big)(t) \in (J_r(t))^\perp$ for a.e. $t \in \mathbb{T}^n$. Using $N_r = N_{J_r}$,

$$P_{J_r(t)}\big(H(t)\big) = P_{J_r(t)}\Big(\big(P_r H\big)(t)\Big) = \big(P_r H\big)(t) \quad \text{for a.e. } t \in \mathbb{T}^n, \tag{6.1.4}$$

where $P_{J_r(t)}$ are associated projections. The vector function H can be taken to be constant. Then, $\big(P_r H\big)(t)$ is measurable, and by (6.1.4), J_r is measurable.

On the contrary, let J_r be a measurable range function. Then, N_{J_r} is a closed subspace of $\mathscr{H}^r$ and consequently $V_r = \mathscr{T}_r^{-1} N_{J_r}$ is closed, SI and according to Lemma 6.1.4 the space V_r clearly satisfies (6.1.2). By Lemma 6.1.5, it is obvious that the correspondence between V_r and J_r is one-to-one. □

Therefore, by Theorem 6.1.1, for every SI space V_r there is the range function J_r such that $V_r = \mathscr{T}_r^{-1} N_{J_r}$. Moreover, for every range function J_r there is the SI space V_r so that $N_{J_r} = \mathscr{T}_r V_r$. It is said that the range function J_r corresponds to V_r or associated to V_r if $V_r = \mathscr{T}_r^{-1} N_{J_r}$, i.e. $N_{J_r} = \mathscr{T}_r V_r$.

Corollary 6.1.1 ([6]). *If J_r is a range function (not necessarily measurable), then there is a unique measurable range function J_r' so that $J_r'(t) \subseteq J_r(t)$ for a.e. $t \in \mathbb{T}^n$, and $N_{J_r'} = N_{J_r}$.*

An additional relation between the range function J_r and the corresponding SI space V_r is given in the next assertion.

Proposition 6.1.1 ([7]). *Let $V_r, U_r \subset H^r$ be SI spaces and J_{V_r}, J_{U_r} be associated range functions, respectively.*

(1) *Let $V_r^{\perp}$ be the orthogonal complement of V_r. Then, $V_r^{\perp}$ is also a SI space and $J_{V_r^{\perp}}(t) = (J_{V_r}(t))^{\perp}$ for a.e. $t \in \mathbb{T}^n$.*

(2) *If $J_{V_r}(t) = J_{U_r}(t)$ for a.e. $t \in \mathbb{T}^n$, then $V_r = U_r$.*

(3) *The space $V_r \cap U_r$ is a SI space with the associated range function $J_{V_r \cap U_r}(t) = J_{V_r}(t) \cap J_{U_r}(t)$ for a.e. $t \in \mathbb{T}^n$.*

Proof. (1) Let $f \in V_r^{\perp}$. Then, $f \notin V_r$ and $T_q f \notin V_r$ for all $q \in \mathbb{Z}^n$. Thus, $T_q f \in V_r^{\perp}$ for all $q \in \mathbb{Z}^n$. Hence, $V_r^{\perp}$ is also a SI space. Further, let $H \in \mathscr{H}^r$. Then, $H(t) \notin J_{V_r}(t)$ if and only if $H(t) \in (J_{V_r}(t))^{\perp}$, for a.e. $t \in \mathbb{T}^n$. Using Theorem 6.1.1, it follows that $H(t) \notin J_{V_r}(t)$ if and only if $\mathscr{T}_r^{-1}H \notin V_r$ if and only if $H(t) \in J_{V_r^{\perp}}(t)$, for a.e. $t \in \mathbb{T}^n$. Therefore, $J_{V_r^{\perp}}(t) = (J_{V_r}(t))^{\perp}$ for a.e. $t \in \mathbb{T}^n$.

(2) Let $J_{V_r}(t) = J_{U_r}(t)$ for a.e. $t \in \mathbb{T}^n$. Then, $N_{J_{V_r}} = N_{J_{U_r}}$ and thus $V_r = U_r$, because $\mathscr{T}_r V_r = N_{J_{V_r}}$ and $\mathscr{T}_r U_r = N_{J_{U_r}}$.

(3) The first part of the statement is obvious. Let $H \in \mathscr{H}^r$. Using Theorem 6.1.1, it follows that

$$\begin{aligned} H(t) \in J_{V_r \cap U_r}(t) \text{ for a.e. } t \in \mathbb{T}^n \quad &\Leftrightarrow \quad \mathscr{T}_r^{-1}H \in V_r \cap U_r \\ &\Leftrightarrow \quad \mathscr{T}_r^{-1}H \in V_r \wedge \mathscr{T}_r^{-1}H \in U_r \\ &\Leftrightarrow \quad H(t) \in J_{V_r}(t) \wedge H(t) \in J_{U_r}(t) \text{ for a.e. } t \in \mathbb{T}^n \\ &\Leftrightarrow \quad H(t) \in J_{V_r}(t) \cap J_{U_r}(t) \text{ for a.e. } t \in \mathbb{T}^n. \end{aligned}$$

Hence, $J_{V_r \cap U_r}(t) = J_{V_r}(t) \cap J_{U_r}(t)$ for a.e. $t \in \mathbb{T}^n$. □

The next definition is Definition 1.0.4 adapted to the observed spaces.

Definition 6.1.4 ([6]). *Let $V_r = \mathscr{T}_r^{-1}N_{J_r}$, where J_r is a given range function.*

(1) *A mapping $\dim_{V_r} : \mathbb{T}^n \to \mathbb{N} \cup \{0, +\infty\}$ defined by $\dim_{V_r}(t) = \dim J_r(t)$ is called the dimension function of V_r.*

(2) *The spectrum of space V_r is defined by $\sigma_{V_r} = \big\{t \in \mathbb{T}^n : \dim J_r(t) > 0\big\}$ or equivalently $\sigma_{V_r} = \big\{t \in \mathbb{T}^n : J_r(t) \neq \{\mathbf{0}\}\big\}$.*

In the continuation, a positive constant will always be denoted by C, and it will be clear from the context whether it is the same constant or not. The Lebesgue measure of a measurable set Q will be denoted by $m(Q)$.

6.2 Characterization of frames

For the characterization of frames, the following auxiliary assertion is very important.

Lemma 6.2.1 ([6]). (1) *If $E_r(\mathscr{A}_{r,I})$ is a Bessel family, then*

$$\sum_{f\in\mathscr{A}_{r,I}}\sum_{q\in\mathbb{Z}^n}\left|\langle T_q f,\varphi\rangle_{H^r}\right|^2=\sum_{f\in\mathscr{A}_{r,I}}\int_{\mathbb{T}^n}\left|\left\langle \mathscr{T}_r f(t),\mathscr{T}_r\varphi(t)\right\rangle_{\ell^2_r}\right|^2\mathrm{d}t$$

for every $\varphi\in\mathscr{A}_{r,I}$.

(2) *Let $g_1,g_2\in L^2$ and $\widehat{f_1}=\widehat{g_1}\mu_{-r}$, $\widehat{f_2}=\widehat{g_2}\mu_{-r}$. Then,*

$$\langle T_q f_1,f_2\rangle_{H^r}=\langle T_q g_1,g_2\rangle_{L^2},\quad q\in\mathbb{Z}^n.$$

Proof. (1) Let $E_r(\mathscr{A}_{r,I})$ be a Bessel family. Then, using the equality (4.4.1), it follows that

$$\begin{aligned}\sum_{f\in\mathscr{A}_{r,I}}\sum_{q\in\mathbb{Z}^n}\left|\langle T_q f,\varphi\rangle_{H^r}\right|^2&=\sum_{f\in\mathscr{A}_{r,I}}\sum_{q\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n}\mathrm{e}^{-2\pi i\langle t,q\rangle}\,\widehat{f}(t)\overline{\widehat{\varphi}(t)}\mu_r^2(t)\,\mathrm{d}t\right|^2\\&=\sum_{g\in\mathscr{A}_I}\sum_{q\in\mathbb{Z}^n}\left|\int_{\mathbb{R}^n}\mathrm{e}^{-2\pi i\langle t,q\rangle}\,\widehat{g}(t)\overline{\widehat{\phi}(t)}\,\mathrm{d}t\right|^2\\&=\sum_{g\in\mathscr{A}_I}\sum_{q\in\mathbb{Z}^n}\left|\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n}\mathrm{e}^{-2\pi i\langle t,q\rangle}\,\widehat{g}(t+k)\overline{\widehat{\phi}(t+k)}\,\mathrm{d}t\right|^2\\&=\sum_{g\in\mathscr{A}_I}\sum_{q\in\mathbb{Z}^n}\left|\int_{\mathbb{T}^n}\mathrm{e}^{-2\pi i\langle t,q\rangle}\sum_{k\in\mathbb{Z}^n}\widehat{g}(t+k)\overline{\widehat{\phi}(t+k)}\,\mathrm{d}t\right|^2\\&=\sum_{g\in\mathscr{A}_I}\int_{\mathbb{T}^n}\left|\sum_{k\in\mathbb{Z}^n}\widehat{g}(t+k)\overline{\widehat{\phi}(t+k)}\right|^2\mathrm{d}t\\&=\sum_{f\in\mathscr{A}_{r,I}}\int_{\mathbb{T}^n}\left|\left\langle \mathscr{T}_r f(t),\mathscr{T}_r\varphi(t)\right\rangle_{\ell^2_r}\right|^2\mathrm{d}t,\end{aligned}$$

where $\widehat{f}\mu_r=\widehat{g}\in L^2$ and $\widehat{\varphi}\mu_r=\widehat{\phi}\in L^2$.

(2) Since $g_1,g_2\in L^2$, using Theorem 3.2.2 and Plancherel's formula (3.1.4), it follows that

$$\begin{aligned}\langle T_q f_1,f_2\rangle_{H^r}&=\int_{\mathbb{R}^n}\widehat{T_q f_1}(t)\overline{\widehat{f_2}(t)}\mu_r^2(t)\,\mathrm{d}t=\int_{\mathbb{R}^n}\mathrm{e}^{-2\pi i\langle t,q\rangle}\,\widehat{f_1}(t)\overline{\widehat{f_2}(t)}\mu_r^2(t)\,\mathrm{d}t\\&=\int_{\mathbb{R}^n}\mathrm{e}^{-2\pi i\langle t,q\rangle}\,\widehat{g_1}(t)\overline{\widehat{g_2}(t)}\,\mathrm{d}t=\int_{\mathbb{R}^n}\widehat{T_q g_1}(t)\overline{\widehat{g_2}(t)}\,\mathrm{d}t=\langle\widehat{T_q g_1},\widehat{g_2}\rangle_{L^2}=\langle T_q g_1,g_2\rangle_{L^2}\end{aligned}$$

for every $q\in\mathbb{Z}^n$. □

The connection of frames in the observed function spaces and frames in the spaces of weighted sequences is given in the next statement.

Theorem 6.2.1 ([6]). *Let $V_r = S_r(\mathscr{A}_{r,I})$. Then, $E_r(\mathscr{A}_{r,I})$ is*

(1) *a frame of V_r with frame bounds A and B if and only if $\{\mathscr{T}_r f(t) : f \in \mathscr{A}_{r.I}\} \subset \ell^2_r$ is a frame of $J_r(t)$ with frame bounds A and B for a.e. $t \in \mathbb{T}^n$;*

(2) *a Riesz family (basis) of V_r with bounds A and B if and only if $\{\mathscr{T}_r f(t) : f \in \mathscr{A}_{r,I}\} \subset \ell^2_r$ is a Riesz family (basis) of $J_r(t)$ with bounds A and B for a.e. $t \in \mathbb{T}^n$;*

(3) *a Bessel family of V_r with bound B if and only if $\{\mathscr{T}_r f(t) : f \in \mathscr{A}_{r,I}\} \subset \ell^2_r$ is a Bessel family of $J_r(t)$ with bound B for a.e. $t \in \mathbb{T}^n$;*

(4) *a fundamental frame of V_r if and only if $\{\mathscr{T}_r f(t) : f \in \mathscr{A}_{r,I}\} \subset \ell^2_r$ is a fundamental frame of $J_r(t)$ for a.e. $t \in \mathbb{T}^n$.*

Proof. The assertions are valid based on the lemmas 6.1.2, 6.1.3, 6.2.1 and on Theorem 1.0.2. □

According to Theorem 6.2.1, the problem of checking whether $E_r(\mathscr{A}_{r,I})$ is a frame or a Riesz family or a Bessel family or a fundamental frame on a "large" subspaces of H^r is reduced to the problem of checking it on a "small" subspaces of ℓ^2_r.

Also, in [6], the characterization of frames was done using the Gram matrix. Therefore, let us introduce the definition of the Gram matrix.

Set

$$\gamma_r^k = (\gamma_r^k(q))_{q\in\mathbb{Z}^n} \subset \ell^2_r, \quad k \in I, \tag{6.2.1}$$

where $\gamma_r^k(q)$ is defined by $\gamma_r^k(q) = \frac{\widehat{g}_k(t+q)}{\mu_r(q)}$ for fixed $t \in \mathbb{T}^n$, and $g_k \in L^2$ such that $\widehat{f_k} = \widehat{g}_k \mu_{-r}$, $k \in I$. Suppose that $(\gamma_r^k)_{k\in I}$ is given. One defines an operator D_r by

$$D_r\alpha = \left(\sum_{k\in I} \alpha_k \gamma_r^k(q)\right)_{q\in\mathbb{Z}^n}, \tag{6.2.2}$$

where $\alpha = (\alpha_k)_{k\in I}$ is a sequence with compact support (only a finite number of elements are non-zero). If the mapping D_r is extended as a continuous mapping $D_r : \ell^2(I) \to \ell^2_r$, then its adjoint operator is given by $D_r^* : \ell^2_r \to \ell^2(I)$,

$$D_r^*\beta = \left(\langle \beta, \gamma_r^k \rangle_{\ell^2_r}\right)_{k\in I}, \quad \beta = (\beta_q)_{q\in\mathbb{Z}^n} \in \ell^2_r. \tag{6.2.3}$$

It is not difficult to see that: D_r is continuous if and only if D_r^* is continuous if and only if $\{\gamma_r^k : k \in I\}$ is a Bessel family. Therefore, $\{\gamma_r^k : k \in I\}$ is a Bessel family with the bound B if $\|D_r^*\|^2 \leqslant B$.

Definition 6.2.1 ([6, 23]). (1) *The mapping $G_r = D_r^* D_r : \ell^2(I) \to \ell^2(I)$ is called the Gramian of $\{\gamma_r^k : k \in I\}$.*

(2) *The mapping $G_r^* = D_r D_r^* : \ell^2_r \to \ell^2_r$ is called the dual Gramian of $\{\gamma_r^k : k \in I\}$.*

Note that, in addition to the Gramian, the name Gram's matrix is also used. By Definition 6.2.1, the following assertion is obvious.

Lemma 6.2.2 ([6]). *The Gramian G_r and the dual Gramian G_r^* are self-adjoint and $\|D_r\|^2 = \|D_r^*\|^2 = \|G_r\| = \|G_r^*\|$.*

A matrix notation of the Gramian is given in Lemma 6.2.3.

Lemma 6.2.3 ([6])**.** *Let $t \in \mathbb{T}^n$ be fixed and $\{\gamma_r^k : k \in I\}$ be given by (6.2.1). Then, the Gramian G_r can be written as a matrix with*

$$G_r(t) = \left[\left\langle \mathscr{T}_r f_k(t), \mathscr{T}_r f_j(t)\right\rangle_{\ell_r^2}\right]_{k,j\in I} = \left[\sum_{q\in\mathbb{Z}^n} \widehat{g_k}(t+q)\overline{\widehat{g_j}(t+q)}\right]_{k,j\in I},$$

and the corresponding dual Gramian with

$$G_r^*(t) = \left[\sum_{k\in I} \frac{\widehat{g_k}(t+q)}{\mu_r(q)} \cdot \frac{\overline{\widehat{g_k}(t+p)}}{\mu_r(p)}\right]_{q,p\in\mathbb{Z}^n}.$$

Proof. Suppose that $\{e_k : k \in I\}$ and $\{\widetilde{e_q} : q \in \mathbb{Z}^n\}$ are the standard basis of $\ell^2(I)$ and ℓ_r^2, respectively. Since $\langle G_r e_k, e_j\rangle_{\ell^2} = \langle D_r e_k, D_r e_j\rangle_{\ell_r^2} = \langle \gamma_r^k, \gamma_r^j\rangle_{\ell_r^2}$, $k, j \in I$, and $\langle G_r^* e_q, e_p\rangle_{\ell_r^2} = \langle D_r^* e_q, D_r^* e_p\rangle_{\ell^2} = \sum_{k\in I} \gamma_r^k(q)\overline{\gamma_r^k(p)}$, $q, p \in \mathbb{Z}^n$, the assertion follows. □

Characterizations of frames and Riesz families via Gram's and dual Gram's matrix are given in the following theorem. Note, the spectrum of an operator P will be denoted by $\sigma(P)$.

Theorem 6.2.2 ([6])**.** *Let $V_r = S_r(\mathscr{A}_{r,I})$. The family $E_r(\mathscr{A}_{r,I})$ is*

1. *a Bessel family of V_r with the bound B if and only if $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|G_r(t)\|_{\ell^2} \leqslant B$ if and only if $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|G_r^*(t)\|_{\ell_r^2} \leqslant B$;*
2. *a frame of V_r with frame bounds A and B if and only if*
$$A\|\beta\|_{\ell_r^2}^2 \leqslant \langle G_r^*(t)\beta, \beta\rangle_{\ell_r^2} \leqslant B\|\beta\|_{\ell_r^2}^2, \tag{6.2.4}$$
where $\beta \in \operatorname{span}\{\mathscr{T}_r f_k(t) : f_k \in \mathscr{A}_{r,I},\ k \in I\}$ for a.e. $t \in \mathbb{T}^n$ if and only if
$$\sigma(G_r^*(t)) \subseteq \{0\} \cup [A, B] \quad \text{for a.e. } t \in \mathbb{T}^n; \tag{6.2.5}$$
3. *a fundamental frame of V_r with frame bounds A and B if and only if $\sigma(G_r^*(t)) \subseteq [A, B]$ for a.e. $t \in \mathbb{T}^n$;*
4. *a Riesz family of V_r with bounds A and B if and only if*
$$A\|\alpha\|_{\ell^2}^2 \leqslant \langle G_r(t)\alpha, \alpha\rangle_{\ell^2} \leqslant B\|\alpha\|_{\ell^2}^2 \quad \text{for a.e. } t \in \mathbb{T}^n,\ \alpha \in \ell^2(I), \tag{6.2.6}$$
if and only if
$$\sigma(G_r(t)) \subseteq [A, B] \quad \text{for a.e. } t \in \mathbb{T}^n; \tag{6.2.7}$$
5. *a Riesz basis of V_r if and only if (6.2.7) holds and $0 \notin \sigma(G_r^*(t))$ for a.e. $t \in \mathbb{T}^n$.*

Proof. The proof of the statement is similar to the proof of the corresponding theorem in [23] for $r = 0$. The statement under (1) follows from Theorem 6.2.1 and the lemmas 6.2.2 and 6.2.3. By

$$\langle G_r^*(t)\beta, \beta\rangle_{\ell_r^2} = \langle D_r^*\beta, D_r^*\beta\rangle_{\ell^2} = \sum_{k\in I} \left|\langle \beta, \gamma_r^k\rangle_{\ell_r^2}\right|^2, \quad \beta \in \ell_r^2,$$

and Theorem 6.2.1, the first equivalence follows. Since by Lemma 6.2.2, $G_r^*(t)$ is self-adjoint, it follows that

$$\ell_r^2 = \ker G_r^*(t) \oplus \overline{\operatorname{rank} G_r^*(t)}.$$

Let J_r be the range function of V_r. Then, $\operatorname{rank} G_r^*(t) = J_r(t)$ for a.e. $t \in \mathbb{T}^n$, because $\ker G_r^*(t) = \ker D_r^* = (J_r(t))^\perp$ for a.e. $t \in \mathbb{T}^n$. Therefore, observing the restriction of $G_r^*(t)$ to $J_r(t)$ yields to the equivalence between (6.2.4) and (6.2.5). Moreover, $E_r(\mathscr{A}_{r,I})$ is a fundamental frame of V_r if $\ker G_r^*(t) = \{\mathbf{0}\}$ for a.e. $t \in \mathbb{T}^n$. Further, by

$$\langle G_r\alpha, \alpha\rangle_{\ell^2} = \langle D_r\alpha, D_r\alpha\rangle_{\ell^2_r} = \left\| \left(\sum_{k\in I} \alpha_k \gamma_r^k(q) \right)_{q\in\mathbb{Z}^n} \right\|^2_{\ell^2_r}, \quad \alpha = (\alpha_k)_{k\in I} \in \ell^2(I),$$

and Theorem 6.2.1, the first equivalence under (4) follows. It is not difficult to see that the operator G_r is non-negative definite. Thus, the equivalence (6.2.6)⇔(6.2.7) follows. Moreover, $E_s(\mathscr{A}_{I,s})$ is a Riesz basis if $\ker G_r^*(t) = \{\mathbf{0}\}$ for a.e. $t \in \mathbb{T}^n$. □

6.3 The decomposition of shift-invariant spaces

De Boor et al. in [22] claimed a statement about the decomposition of finitely generated SI spaces in quasi-regular spaces. Then, Marcin Bownik proved the decomposition theorem for SI subspaces of L^2 in [23]. In this dissertation, the decomposition theorem of SI subspaces of the Sobolev space H^r is being proved.

Definition 6.3.1 ([5, 7])**.** *A set $\mathscr{A}_{r,m}$ is said to be a frame generator for $S_r(\mathscr{A}_{r,m})$ if their integer translations form a frame for $S_r(\mathscr{A}_{r,m})$, i.e. if $E_r(\mathscr{A}_{r,m})$ is a frame for $S_r(\mathscr{A}_{r,m})$.*

Definition 6.3.2 ([6, 23])**.** *A function $f_0 \in V_r = S_r(f)$, $f \in H^r$, is called a tight frame (or quasi-orthogonal) generator of V_r if for all $\varphi \in V_r$ holds*

$$\|\varphi\|^2_{H^r} = \sum_{q\in\mathbb{Z}^n} \left|\langle T_q f_0, f\rangle_{H^r}\right|^2.$$

To prove the decomposition theorem, the following auxiliary assertion is necessary.

Lemma 6.3.1 ([6, 22, 23])**.** *The function $f_0 \in V_r = S_r(f)$ is a tight frame generator of V_r if and only if $\|\mathscr{T}_r f_0(t)\|_{\ell^2_r} = \mathbf{1}_{\sigma_{V_r}}(t)$ for a.e. $t \in \mathbb{T}^n$.*

Proof. By the theorems 6.1.1 and 6.2.1, the assertion follows. □

Note, unless stated otherwise, $\oplus$ will denote the orthogonal sum, and $W = V \oplus U$ or $V = W \ominus U$ will be written.

Theorem 6.3.1 (**The decomposition theorem,** [6])**.** *Let V_r be a SI subspace of H^r. Then, V_r can be decomposed as an orthogonal sum of PSI spaces, i.e.*

$$V_r = \bigoplus_{k\in\mathbb{N}} S_r(f_k), \tag{6.3.1}$$

such that f_k is a tight frame generator of $S_r(f_k)$ and $\sigma_{S_r(f_{k+1})} \subset \sigma_{S_r(f_k)}$ for every $k \in \mathbb{N}$. Moreover, hold:

(1) $\dim_{S_r(f_k)}(t) = \|\mathscr{T}_r f_k(t)\|_{\ell^2_r}$ *for a.e.* $t \in \mathbb{T}^n$, $k \in \mathbb{N}$,

(2) $\dim_{V_r}(t) = \sum_{k\in\mathbb{N}} \|\mathscr{T}_r f_k(t)\|_{\ell^2_r}$ *for a.e.* $t \in \mathbb{T}^n$.

Proof. Let $U_r \subset H^r$ be a SI space with the associated range function J_r and the corresponding projections P_{J_r}. Let $\pi : \mathbb{N} \to \mathbb{Z}^n$ be a bijection. The function $F \in \mathscr{T}_r U_r$ is defined as follows. If $U_r = \{\mathbf{0}\}$, then $F = \mathbf{0}$. Otherwise, let $\gamma_\nu \in \mathscr{H}^r$, $\nu \in \mathbb{N}$, be defined by

$$\gamma_\nu(t) = \begin{cases} \frac{P_{J_r(t)} e_{\pi(\nu)}}{\|P_{J_r(t)} e_{\pi(\nu)}\|_{\ell_r^2}}, & t \in A_\nu, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $A_\nu = \{t \in \mathbb{T}^n : P_{J_r(t)} e_{\pi(\nu)} \neq \mathbf{0}\}$, $\nu \in \mathbb{N}$, and define F by

$$F = \sum_{\nu \in \mathbb{N}} \gamma_\nu \mathbf{1}_{B_\nu},$$

where $B_1 = A_1$, $B_{\nu+1} = A_{\nu+1} \setminus \bigcup_{k=1}^{\nu} A_k$, $\nu \in \mathbb{N}$. Now, $F(t) \in J_r(t)$ and $\|F(t)\|_{\ell_r^2} = \mathbf{1}_{\sigma_{U_r}}(t)$ for a.e. $t \in \mathbb{T}^n$, because $\sigma_{U_r} = \bigcup_{\nu \in \mathbb{N}} A_\nu$. Let $f = \mathscr{T}_r^{-1} F$. Then, by Lemma 6.3.1, f is a tight frame generator of $S_r(f) \subseteq U_r$. Moreover, $\sigma_{S_r(f)} = \sigma_{U_r}$. Obviously,

$$\mathscr{T}_r(U_r \ominus S_r(f)) = \big\{H \in \mathscr{H}^r : H(t) \in J_r(t),\ \langle F(t), H(t)\rangle_{\ell_r^2} = 0 \text{ for a.e. } t \in \mathbb{T}^n\big\}.$$

Further, let $\nu_0 = \min\{\nu \in \mathbb{N} : m(A_\nu) \neq 0\}$. For $H \in \mathscr{T}_r(U_r \ominus S_r(f))$ holds

$$\langle H(t), e_{\pi(\nu)}\rangle_{\ell_r^2} = \langle H(t), P_{J_r(t)} e_{\pi(\nu)}\rangle_{\ell_r^2} = 0 \quad \text{for a.e. } t \in \mathbb{T}^n,\ \nu = 1, \ldots, \nu_0. \tag{6.3.2}$$

Now, by induction on ν, a sequence of tight frame generators can be defined, as follows. Choose $f_1 = \mathscr{T}_r^{-1} F(V_r)$ and assume that $f_1, \ldots, f_\nu$ are constructed for some $\nu \in \mathbb{N}$ such that:

(a) $f_k \in V_r$ is a tight frame generator of $S_r(f_k)$, $k = 1, \ldots, \nu$;

(b) for all $k \neq l$, the spaces $S_r(f_k)$ and $S_r(f_l)$ are disjoint;

(c) if $H \in \mathscr{T}_r V_r^\nu$, then $\langle H(t), e_{\pi(k)}\rangle_{\ell_r^2} = 0$, $k = 1, \ldots, \nu$, for a.e. $t \in \mathbb{T}^n$, where

$$V_r^\nu = V_r \ominus \left(\bigoplus_{k=1}^{\nu} S_r(f_k)\right). \tag{6.3.3}$$

Further, let $f_{\nu+1} = \mathscr{T}_r^{-1} F(V_r^\nu)$. Then, by construction, it follows that the set $\{f_1, \ldots, f_{\nu+1}\}$ satisfies conditions (a)–(c). Indeed, since $\|\mathscr{T}_r f_{\nu+1}(t)\|_{\ell_r^2} = \mathbf{1}_{\sigma_{V_r^\nu}}(t)$, it follows (a); (b) follows from the fact that $S_r(f_{\nu+1}) \subset V_r^\nu$ and by (6.3.3); (*c*) is a consequence of (6.3.2).

Choose $H \in \mathscr{T}_r\big(\bigcap_{k=1}^{+\infty} V_r^k\big)$. Then, using (c), it follows that $\langle H(t), e_{\pi(k)}\rangle_{\ell_r^2} = 0$, $k \in \mathbb{N}$, for a.e. $t \in \mathbb{T}^n$. Hence, $H = \mathbf{0}$ and thus $\bigcap_{k=1}^{+\infty} V_r^k = \{\mathbf{0}\}$, i.e. (6.3.1) follows. Since $V_r^{k+1} \subset V_r^k$, it leads to $\sigma_{S_r(f_{k+1})} = \sigma_{V_r^{k+1}} \subset \sigma_{V_r^k} = \sigma_{S_r(f_k)}$.

Finally, by (6.3.1), $\dim_{V_r}(t) = \sum_{k \in \mathbb{N}} \|\mathscr{T}_r f_k(t)\|_{\ell_r^2}$ for a.e. $t \in \mathbb{T}^n$. □

Remark 6.3.1. *The decomposition of a SI space $V_r \subset H^r$ is unique only in the case when $\dim_{V_r}(t) \leqslant 1$ for a.e. $t \in \mathbb{T}^n$. If $\operatorname{ess\,sup}_{t \in \mathbb{T}^n} \dim_{V_r}(t) = k_0$ for some $k_0 \in \mathbb{N}$, then the decomposition has k_0 non-trivial components $S_r(f_1)$, …, $S_r(f_{k_0})$ and $S_r(f_k) = \{\mathbf{0}\}$ for $k > k_0$.*

One of the consequences of the decomposition theorem is the following statement. Other consequences will be listed in the following sections.

Proposition 6.3.1 ([7])**.** *Let J_r be the range function associated with the SI space $V_r \subset H^r$ such that* $\dim J_r(t) < +\infty$ *for a.e.* $t \in \mathbb{T}^n$*. Then, there exist* $\{f_k : f_k \in H^r,\, k \in \mathbb{N}\}$ *and measurable sets* $(A_m)_{m\in\mathbb{N}_0}$ *so that* $\bigcup_{m\in\mathbb{N}_0} A_m = \mathbb{T}^n$, $A_k \cap A_j = \emptyset$, $k \neq j$, *and hold:*

(1) $\{T_q f_k : k \in \mathbb{N},\, q \in \mathbb{Z}^n\}$ *is a Parseval frame for* V_r,

(2) *if* $k > m$, *then* $\mathscr{T}_r f_k(t) = \mathbf{0}$ *for a.e.* $t \in A_m$,

(3) *the family* $\{\mathscr{T}_r f_1(t), \dots, \mathscr{T}_r f_m(t)\}$ *is an orthonormal basis for* $J_r(t)$, *for a.e.* $t \in A_m$,

(4) $\dim J_r(t) = m$ *for a.e.* $t \in A_m$.

Proof. Let $\{f_k : f_k \in H^r,\, k \in \mathbb{N}\}$ be the set of functions from Theorem 6.3.1. Then, $\{T_q f_k : k \in \mathbb{N},\, q \in \mathbb{Z}^n\}$ is a Parseval frame for V_r. Moreover, $\{\mathscr{T}_r f_k(t) : k \in \mathbb{N}\}$ is a Parseval frame for $J_r(t)$ for a.e. $t \in \mathbb{T}^n$, by Theorem 6.2.1.

Since $\sigma_{S_r(f_{k+1})} \subset \sigma_{S_r(f_k)}$ for all $k \in \mathbb{N}$, it implies that the family of disjoint sets $(A_m)_{m\in\mathbb{N}_0}$ can be defined as follows:

$$A_0 = \mathbb{T}^n \setminus \sigma_{V_r} \quad \text{and} \quad A_m = \sigma_{S_r(f_m)} \setminus \sigma_{S_r(f_{m+1})} \text{ for } m \in \mathbb{N}.$$

The assumptions $\dim J_r(t) < +\infty$ gives $\sum_{k\in\mathbb{N}} \|\mathscr{T}_r f_k(t)\|_{\ell^2_r} < +\infty$, and thus $\bigcap_{k\in\mathbb{N}} \sigma_{S_r(f_k)} = \emptyset$. Therefore, $\bigcup_{m\in\mathbb{N}_0} A_m = \mathbb{T}^n$.

Now, for $m \in \mathbb{N}$, (2) holds. Using (1), Theorem 6.3.1 and Lemma 6.1.2, (3) follows. Thus, $\dim J_r(t) = m$ for a.e. $t \in A_m$. If $m = 0$, then $J_r(t) = \{\mathbf{0}\}$ for a.e. $t \in A_0$. □

6.4 Shift-preserving and range operators

For further characterization of SI subspaces of the Sobolev space H^r, $r \in \mathbb{R}$, it is necessary to introduce and investigate the relationship between shift-preserving operators and range operators.

Let $V_r \subset H^r$ be a SI space with the corresponding range function J_r and projections P_{J_r}. The definition of shift-preserving operators is given in Introduction, and here it will be formally introduced for the spaces V_r.

Definition 6.4.1 ([7, 23])**.** *A bounded linear operator* $L_r : V_r \to H^r$ *is called a shift-preserving operator if* $L_r T_q = T_q L_r$ *for every* $q \in \mathbb{Z}^n$.

Analogous to Definition 1.0.5, the definition for the observed spaces is introduced.

Definition 6.4.2 ([7])**.** *An operator defined on* J_r *(with values in* ℓ^2_r*) by*

$$R_r : \mathbb{T}^n \to \{\text{bounded operators defined on closed subspaces of } \ell^2_r\},$$

such that the domain of $R_r(t)$ *is* $J_r(t)$ *for a.e.* $t \in \mathbb{T}^n$, *is called the range operator. The range operator* R_r *is measurable if* $t \mapsto R_r(t) P_{J_r(t)}$, $t \in \mathbb{T}^n$, *is a weakly measurable operator.*

Now, following [23] analogous results are obtained for SI spaces $V_r \subset H^r$.

Theorem 6.4.1 ([7])**.** *Assume that* $f \in H^r$ *is a tight frame generator of* $S_r(f)$ *and* $L_r : S_r(f) \to H^r$ *is a shift-preserving operator. Let* $F = \mathscr{T}_r f$. *Then,*

$$\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)(\omega F)(t) = \omega(t)\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big) F(t) \tag{6.4.1}$$

for a.e. $t \in \mathbb{T}^n$, $\omega \in L^2_r(\mathbb{T}^n)$.

Proof. Let the assumptions given in the theorem hold. By Lemma 6.1.2,

$$\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)(M_{-q}F) = \big(\mathscr{T}_r L_r T_q\big) f = \big(\mathscr{T}_r T_q L_r\big)(\mathscr{T}_r^{-1}\mathscr{T}_r f) = M_{-q}\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)F, \quad q \in \mathbb{Z}^n.$$

Therefore, using linearity, for all polynomials

$$p_k(t) = \sum_{|q|\leqslant k} \alpha_q \, \mathrm{e}^{-2\pi i\langle t,q\rangle}, \quad k \in \mathbb{N},$$

holds

$$\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)(p_k F)(t) = p_k(t)\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)F(t) \quad \text{for a.e. } t \in \mathbb{T}^n. \tag{6.4.2}$$

Using the lemmas 6.1.2 and 6.3.1 and boundedness of L_r, it follows that

$$\begin{aligned}
\int_{\mathbb{T}^n} |p_k(t)|^2 \big\|(\mathscr{T}_r L_r \mathscr{T}_r^{-1})F(t)\big\|_{\ell_r^2}^2 \,\mathrm{d}t &= \int_{\mathbb{T}^n} \big\|(\mathscr{T}_r L_r \mathscr{T}_r^{-1})(p_k F)(t)\big\|_{\ell_r^2}^2 \,\mathrm{d}t \\
&= \big\|(\mathscr{T}_r L_r \mathscr{T}_r^{-1})(p_k F)\big\|_{\mathscr{H}^r}^2 \\
&\leqslant C\big\|p_k F\big\|_{\mathscr{H}^r}^2 \\
&= C \int_{\mathbb{T}^n} |p_k(t)|^2 \|F(t)\|_{\ell_r^2}^2 \,\mathrm{d}t \\
&= C \int_{\mathbb{T}^n} |p_k(t)|^2 \mathbf{1}_{\sigma_{S_r(f)}}(t) \,\mathrm{d}t < +\infty.
\end{aligned} \tag{6.4.3}$$

It is known (by Lusin's theorem) that for every function $\psi \in L^\infty(\mathbb{T}^n)$ there is a sequence of polynomials $(p_{k_\nu}^\nu)_{\nu\in\mathbb{N}}$ so that:

$$\|p_{k_\nu}^\nu\|_\infty \leq \|\psi\|_\infty, \, \nu \in \mathbb{N}, \quad \text{and} \quad \lim_{\nu\to+\infty} p_{k_\nu}^\nu(t) = \psi(t) \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Thus, using the Lebesgue Dominated Convergence Theorem, (6.4.3) leads to

$$\int_{\mathbb{T}^n} |\psi(t)|^2 \big\|(\mathscr{T}_r L_r \mathscr{T}_r^{-1})F(t)\big\|_{\ell_r^2}^2 \,\mathrm{d}t \leqslant C \int_{\mathbb{T}^n} |\psi(t)|^2 \|F(t)\|_{\ell_r^2}^2 \,\mathrm{d}t,$$

and so

$$\big\|(\mathscr{T}_r L_r \mathscr{T}_r^{-1})F(t)\|_{\ell_r^2} \leqslant C\|F(t)\|_{\ell_r^2} \quad \text{for a.e. } t \in \mathbb{T}^n. \tag{6.4.4}$$

Finally, take a sequence of polynomials $(p_{k_\nu}^\nu)_{\nu\in\mathbb{N}}$ so that $p_{k_\nu}^\nu \to \omega$ in $L_r^2(\mathbb{T}^n)$ and

$$\lim_{\nu\to+\infty} p_{k_\nu}^\nu(t) = \omega(t), \quad \lim_{\nu\to+\infty} \big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)(p_{k_\nu}^\nu F)(t) = \big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)(\omega F)(t), \tag{6.4.5}$$

for a.e. $t \in \mathbb{T}^n$. From (6.4.2), using (6.4.4) and (6.4.5), it follows that (6.4.1) holds for every $\omega \in L_r^2(\mathbb{T}^n)$. □

An immediate consequence of the theorems 6.1.1 and 6.4.1 is Corollary 6.4.1.

Corollary 6.4.1 ([7])**.** *Assume that $V_r \subset H^r$ is a SI space and $L_r : V_r \to H^r$ is a shift-preserving operator. Let $F \in \mathscr{T}_r V_r$ and let ω be a measurable function such that $\omega F \in \mathscr{H}^r$ (and therefore $\omega F \in \mathscr{T}_r V_r$). Then,*

$$\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)(\omega F)(t) = \omega(t)\big(\mathscr{T}_r L_r \mathscr{T}_r^{-1}\big)F(t) \quad \textit{for a.e. } t \in \mathbb{T}^n.$$

The basic connection between the shift-preserving operator and the range operator is given by the following theorem.

Theorem 6.4.2 ([7]). *Assume that $V_r \subset H^r$ is a SI space and J_r is the associated range function.*

(1) *If $L_r : V_r \to H^r$ is a shift-preserving operator, then there is a measurable range operator R_r on J_r so that*

$$(\mathscr{T}_r L_r)f(t) = R_r(t)(\mathscr{T}_r f(t)) \quad \text{for a.e. } t \in \mathbb{T}^n,\ f \in V_r. \tag{6.4.6}$$

(2) *If R_r is a measurable range operator on J_r so that $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|R_r(t)\| < +\infty$, then there is a shift-preserving operator $L_r : V_r \to H^r$ so that* (6.4.6) *holds.*

The correspondence between L_r and R_r is one-to-one and $\operatorname{ess\,sup}_{t\in\mathbb{T}^n} \|R_r(t)\| = \|L_r\|$.

Proof. (1) First, by Theorem 6.3.1, $V_r = \oplus_{k\in\mathbb{N}} S_r(f_k)$ where f_k is a tight frame generator of $S_r(f_k)$. Now, $V_r^\nu = \oplus_{k=1}^{\nu} S_r(f_k)$ with the corresponding range function J_r^ν is observed. Set $F_k = \mathscr{T}_r f_k$. Then, $\{F_1(t), ..., F_\nu(t)\} \setminus \{\mathbf{0}\}$ is an orthonormal basis of $J_r^\nu(t)$ for a.e. $t \in \mathbb{T}^n$. Note, if $t \notin \sigma_{V_r^\nu}$, then $F_k(t) = \mathbf{0}$, $k = 1, \dots, \nu$. Define the operator $R_r^\nu(t) : J_r^\nu(t) \to \ell_r^2$ by

$$R_r^\nu(t)\bigg(\sum_{k=1}^{\nu} \alpha_k F_k(t)\bigg) = \sum_{k=1}^{\nu} \alpha_k (\mathscr{T}_r L_r \mathscr{T}_r^{-1}) F_k(t),$$

where, $\alpha_k \in \mathbb{C}$, $k = 1, \dots, \nu$. This operator R_r^ν is well defined by (6.4.4). Now, for each $\varphi \in V_r^\nu$ there exists $\varphi_k \in S_r(f_k)$, $k = 1, ..., \nu$, so that $\varphi = \varphi_1 + \cdots + \varphi_\nu$ and

$$\mathscr{T}_r\varphi = \mathscr{T}_r\varphi_1 + \cdots + \mathscr{T}_r\varphi_\nu = \omega_1 F_1 + \cdots + \omega_\nu F_\nu$$

for some $\omega_k \in L_r^2(\mathbb{T}^n)$. Thus,

$$\begin{aligned}(\mathscr{T}_r L_r)\varphi(t) &= (\mathscr{T}_r L_r \mathscr{T}_r^{-1})\bigg(\sum_{k=1}^{\nu} \omega_k F_k\bigg)(t) = \sum_{k=1}^{\nu} \omega_k(t) (\mathscr{T}_r L_r \mathscr{T}_r^{-1}) F_k(t) \\ &= \sum_{k=1}^{\nu} \omega_k(t) R_r^\nu(t)\big(F_k(t)\big) = \sum_{k=1}^{\nu} R_r^\nu(t)\big(\omega_k(t) F_k(t)\big) \\ &= R_r^\nu(t)\big(\mathscr{T}_r\varphi(t)\big),\end{aligned} \tag{6.4.7}$$

by Theorem 6.4.1. Since $\mathscr{T}_r$ is an isometry, it implies that R_r^ν is a measurable operator. Since L_r is a shift-preserving operator, it is bounded i.e. $\|L_r\| \leqslant C$. In order to prove $\|R_r^\nu(t)\| \leqslant C$ for a.e. $t \in \mathbb{T}^n$, one must first prove

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_r^\nu(t)(\Theta_a(t))\|_{\ell_r^2} \leqslant C, \tag{6.4.8}$$

where $\Theta_a \in \mathscr{H}^r$ is given by

$$\Theta_a(t) = \sum_{k=1}^{\nu} a_k F_k(t)$$

for $a = (a_1, \dots, a_\nu) \in \mathbb{S}^{\nu-1}$, $\mathbb{S}^{\nu-1} = \{a \in \mathbb{C}^\nu : |a_1|^2 + \cdots + |a_\nu|^2 = 1\}$. It is not difficult to see that $\|\Theta_a(t)\|_{\ell_r^2} = 1$.

Assume that (6.4.8) is false. Then, there are $\varepsilon > 0$ and a measurable set $Q \subset \mathbb{T}^n$, whose measure is not zero, so that $\|R_r^\nu(t)(\Theta_a(t))\|_{\ell_r^2} > C + \varepsilon$ for $t \in Q$. Set $\Theta = \Theta_a \mathbf{1}_Q$ and $\theta = \mathscr{T}_r^{-1}\Theta \in V_r^\nu$. Then,

$$\|(\mathscr{T}_r L_r)\theta\|_{\mathscr{H}^r} = \|L_r\theta\|_{H^r} \leqslant C\|\theta\|_{H^r} = C\|\Theta\|_{\mathscr{H}^r},$$

because $\mathscr{T}_r$ is an isometry and $\|L_r\| \leqslant C$. However, using (6.4.7),

$$\begin{aligned}\|(\mathscr{T}_r L_r)\theta\|^2_{\mathscr{H}^r} &= \int_{\mathbb{T}^n} \|R_r^\nu(t)(\Theta(t))\|^2_{\ell_r^2}\,\mathrm{d}t = \int_Q \|R_r^\nu(t)(\Theta_a(t))\|^2_{\ell_r^2}\,\mathrm{d}t \\ &\geqslant (C+\varepsilon)^2 \int_Q \mathrm{d}t = (C+\varepsilon)^2 \int_Q \|\Theta_a(t)\|^2_{\ell_r^2}\,\mathrm{d}t = (C+\varepsilon)^2\|\Theta\|^2_{\mathscr{H}^r},\end{aligned}$$

which is impossible. Hence, (6.4.8) is true. Finally, for a dense subset $(a_m)_{m\in\mathbb{N}}$ of $\mathbb{S}^{\nu-1}$,

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_r^\nu(t)\| = \operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \sup_{a\in\mathbb{S}^{\nu-1}} \|R_r^\nu(t)(\Theta_a(t))\|_{\ell_r^2} = \operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \sup_{m\in\mathbb{N}} \|R_r^\nu(t)(\Theta_{a_m}(t))\|_{\ell_r^2} \leqslant C,$$

by (6.4.8). Therefore, $\|R_r^\nu(t)\| \leqslant C$ for a.e. $t \in \mathbb{T}^n$.

Let $j \leqslant \nu$ and note that $R_r^j(t) = R_r^\nu(t)|_{J_r^j(t)}$. Define $R_r(t) : \bigcup_{j\in\mathbb{N}} J_r^j(t) \to \ell_r^2$ by $R_r(t)(\alpha) = R_r^j(t)(\alpha)$, $\alpha \in J_r^j(t)$, for some $j \in \mathbb{N}$. Then,

$$\|R_r(t)(\alpha)\|_{\ell_r^2} \leqslant C\|\alpha\|_{\ell_r^2}, \quad \alpha \in \bigcup_{j\in\mathbb{N}} J_r^j(t),$$

since $\|R_r^\nu(t)\| \leqslant C$ for a.e. $t \in \mathbb{T}^n$. Since

$$J_r(t) = \overline{\bigcup_{j\in\mathbb{N}} J_r^j(t)},$$

it follows that $R_r(t)$ can uniquely be extended to $R_r(t) : J_r(t) \to \ell_r^2$ with $\|R_r(t)\| \leqslant C$. Now, (6.4.6) holds. Indeed, choose $f \in V_r$ and a sequence $(f_\nu)_{\nu\in\mathbb{N}}$, $f_\nu \in V_r^\nu$, so that

$$\lim_{\nu\to+\infty} f_\nu = f \text{ in } H^r, \quad \lim_{\nu\to+\infty} \mathscr{T}_r f_\nu(t) = \lim_{\nu\to+\infty} \mathscr{T}_r f(t) \quad \text{and}$$
$$\lim_{\nu\to+\infty} (\mathscr{T}_r L_r) f_\nu(t) = \lim_{\nu\to+\infty} (\mathscr{T}_r L_r) f(t)$$

for a.e. $t \in \mathbb{T}^n$. Then, by (6.4.7) and the previous construction, it follows that

$$(\mathscr{T}_r L_r) f_\nu(t) = R_r(t)(\mathscr{T}_r f_\nu(t)) \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Letting $\nu \to +\infty$ gives (6.4.6).

(2) Assume that R_r is a measurable range operator on J_r so that

$$\operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_r(t)\| = C < +\infty.$$

Then, $R_r(t)(\mathscr{T}_r f(t))$ is measurable for a.e. $t \in \mathbb{T}^n$, $f \in V_r$, and

$$\begin{aligned}\|R_r(\mathscr{T}_r f)\|^2_{\mathscr{H}^r} &= \int_{\mathbb{T}^n} \|R_r(t)(\mathscr{T}_r f(t))\|^2_{\ell_r^2}\,\mathrm{d}t \leqslant \operatorname*{ess\,sup}_{t\in\mathbb{T}^n} \|R_r(t)\|^2 \int_{\mathbb{T}^n} \|\mathscr{T}_r f(t)\|^2_{\ell_r^2}\,\mathrm{d}t \\ &= C^2\|\mathscr{T}_r f\|^2_{\mathscr{H}^r} = C^2\|f\|^2_{H^r}, \quad f \in V_r. \end{aligned} \tag{6.4.9}$$

Define $L_r : V_r \to H^r$ by $L_r f = \mathscr{T}_r^{-1} R_r(\mathscr{T}_r f)$. Then, L_r is linear and satisfies (6.4.6); by (6.4.9), $\|L_r f\|_{H^r} \leqslant C\|f\|_{H^r}$. Moreover, L_r is shift-preserving, since

$$\begin{aligned}(\mathscr{T}_r L_r T_q)f(t) &= R_r(t)(\mathscr{T}_r T_q f(t)) = R_r(t)(M_{-q}\mathscr{T}_r f(t)) = M_{-q}R_r(t)(\mathscr{T}_r f(t)) \\ &= M_{-q}\mathscr{T}_r(L_r f)(t) = (\mathscr{T}_r T_q L_r)f(t), \quad q \in \mathbb{Z}^n.\end{aligned}$$

Finally, the one-to-one correspondence between R_r and L_r follows from (6.4.6). □

Other properties between the shift-preserving operator and the range operator follow from the basic connection (Theorem 6.4.2).

Theorem 6.4.3 ([7]). *Assume that $V_r \subset H^r$ is a SI space and R_r is the corresponding range operator on J_r. Then,*

$$\|R_r(t)(\alpha)\|_{\ell^2_r} \geqslant C\|\alpha\|_{\ell^2_r}, \quad \alpha \in J_r(t), \text{ for a.e. } t \in \mathbb{T}^n \tag{6.4.10}$$

if and only if

$$\|L_r f\|_{H^r} \geqslant C\|f\|_{H^r}, \quad f \in V_r, \tag{6.4.11}$$

where C is some positive constant.

Proof. Let (6.4.10) hold. Then,

$$\begin{aligned}\|L_r f\|^2_{H^r} &= \|(\mathscr{T}_r L_r)f\|^2_{\mathscr{H}^r} = \int_{\mathbb{T}^n} \|R_r(t)(\mathscr{T}_r f(t))\|^2_{\ell^2_r}\,\mathrm{d}t \geqslant \int_{\mathbb{T}^n} C^2\|\mathscr{T}_r f(t)\|^2_{\ell^2_r}\,\mathrm{d}t \\ &= C^2\|\mathscr{T}_r f\|^2_{\mathscr{H}^r} = C^2\|f\|^2_{H^r}, \quad f \in V_r,\end{aligned}$$

by (6.4.6). Hence, (6.4.11) holds.

Now, let (6.4.11) hold and let $\{\alpha^1, \alpha^2, \ldots\}$ be a dense subset of ℓ^2_r. Then, for a.e. $t \in \mathbb{T}^n$,

$$\|R_r(t)(P_{J_r(t)}(\alpha^k))\|_{\ell^2_r} \geqslant C\|P_{J_r(t)}(\alpha^k)\|_{\ell^2_r}, \quad k \in \mathbb{N}. \tag{6.4.12}$$

Indeed, let (6.4.12) not be valid. Then, there are a measurable set $Q \subset \mathbb{T}^n$ with $m(Q) \neq 0$, $k_0 \in \mathbb{N}$ and $\varepsilon > 0$ so that

$$\|R_r(t)(P_{J_r(t)}(\alpha^{k_0}))\|_{\ell^2_r} \leqslant (C-\varepsilon)\|P_{J_r(t)}(\alpha^{k_0})\|_{\ell^2_r} \quad \text{for } t \in Q.$$

Choose $f \in V_r$ so that $\mathscr{T}_r f(t) = \mathbf{1}_Q(t) P_{J_r(t)}(\alpha^{k_0})$. Then,

$$\begin{aligned}\|L_r f\|^2_{H^r} &= \|(\mathscr{T}_r L_r)f\|^2_{\mathscr{H}^r} = \int_{\mathbb{T}^n} \|(\mathscr{T}_r L_r)f(t)\|^2_{\ell^2_r}\,\mathrm{d}t = \int_{\mathbb{T}^n} \|R_r(t)(\mathscr{T}_r f(t))\|^2_{\ell^2_r}\,\mathrm{d}t \\ &\leqslant (C-\varepsilon)^2 \int_Q \|P_{J_r(t)}(\alpha^{k_0})\|^2_{\ell^2_r}\,\mathrm{d}t = (C-\varepsilon)^2 \int_{\mathbb{T}^n} \|\mathscr{T}_r f(t)\|^2_{\ell^2_r}\,\mathrm{d}t = (C-\varepsilon)^2\|\mathscr{T}_r f\|^2_{\mathscr{H}^r} \\ &= (C-\varepsilon)^2\|f\|^2_{H^r},\end{aligned}$$

which contradicts (6.4.11). Hence, (6.4.10) holds. □

Corollary 6.4.2 ([7]). *Let R_r be the corresponding range operator for a shift-preserving operator $L_r : V_r \to H^r$. Then, $R_r(t)$ is an isometry for a.e. $t \in \mathbb{T}^n$ if and only if L_r is an isometry.*

Proof. Since

$$\|R_r(\mathscr{T}_r f)\|_{\mathscr{H}^r} = \|(\mathscr{T}_r L_r)f\|_{\mathscr{H}^r} = \|L_r f\|_{H^r} \leqslant C\|f\|_{H^r} = C\|\mathscr{T}_r f\|_{\mathscr{H}^r}, \quad f \in V_r,$$

the assertion follows. □

Theorem 6.4.4 ([7]). *Assume that $V_r \subset H^r$ is a SI space, J_r is the associated range function and R_r is the corresponding range operator for a shift-preserving operator $L_r : V_r \to V_r$.*

1. *The adjoint operator $L_r^* : V_r \to V_r$ of L_r is shift-preserving. Moreover, the corresponding range operator is given by $R_r^*(t) = (R_r(t))^*$ for a.e. $t \in \mathbb{T}^n$.*
2. *Let $A, B \in \mathbb{R}$ so that $A \leqslant B$. The operator $R_r(t)$ is self-adjoint and $\sigma(R_r(t)) \subseteq [A, B]$ for a.e. $t \in \mathbb{T}^n$ if and only if L_r is a self-adjoint operator and $\sigma(L_r) \subseteq [A, B]$.*
3. *The operator $R_r(t)$ is a unitary operator for a.e. $t \in \mathbb{T}^n$ if and only if L_r is a unitary operator.*

Proof. (1) It is easily seen that R_r^* is a measurable range operator and uniformly bounded on J_r. According to Theorem 6.4.2, there is a shift-preserving operator $L_r^\diamond : V_r \to H^r$ so that $(\mathscr{T}_r L_r^\diamond)f(t) = R_r^*(t)(\mathscr{T}_r f(t))$, $f \in V_r$. Then, for $f, \varphi \in V_r$,

$$\begin{aligned}\langle L_r f, \varphi\rangle_{H^r} &= \langle (\mathscr{T}_r L_r)f, \mathscr{T}_r\varphi\rangle_{\mathscr{H}^r} = \int_{\mathbb{T}^n} \langle (\mathscr{T}_r L_r)f(t), \mathscr{T}_r\varphi(t)\rangle_{\ell_r^2}\,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \langle R_r(t)(\mathscr{T}_r f(t)), \mathscr{T}_r\varphi(t)\rangle_{\ell_r^2}\,\mathrm{d}t = \int_{\mathbb{T}^n} \langle \mathscr{T}_r f(t), R_r^*(t)(\mathscr{T}_r\varphi(t))\rangle_{\ell_r^2}\,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \langle \mathscr{T}_r f(t), (\mathscr{T}_r L_r^\diamond)\varphi(t)\rangle_{\ell_r^2}\,\mathrm{d}t = \langle \mathscr{T}_r f, (\mathscr{T}_r L_r^\diamond)\varphi\rangle_{\mathscr{H}^r} = \langle f, L_r^\diamond\varphi\rangle_{H^r}.\end{aligned}$$

Hence, $L_r^\diamond = L_r^*$.

(2) Using the part (1), it follows that $R_r^*(t) = R_r(t)$ for a.e. $t \in \mathbb{T}^n$ if and only if $L_r^* = L_r$. Suppose that $\sigma(L_r) \subseteq [A, B]$, i.e.

$$A\|f\|_{H^r}^2 \leqslant \int_{\mathbb{T}^n} \langle R_r(t)(\mathscr{T}_r f(t)), \mathscr{T}_r f(t)\rangle_{\ell_r^2}\,\mathrm{d}t \leqslant B\|f\|_{H^r}^2, \quad f \in V_r, \tag{6.4.13}$$

since

$$\langle L_r f, f\rangle_{H^r} = \langle \mathscr{T}_r L_r f, \mathscr{T}_r f\rangle_{\mathscr{H}^r} = \int_{\mathbb{T}^n} \langle R_r(t)(\mathscr{T}_r f(t)), \mathscr{T}_r f(t)\rangle_{\ell_r^2}\,\mathrm{d}t.$$

Now, by similar arguments as in the proof of Theorem 6.4.3, the assertion follows. Therefore, let $\{\alpha^1, \alpha^2, \ldots\}$ be a dense subset of ℓ_r^2. Then, for a.e. $t \in \mathbb{T}^n$ and every $k \in \mathbb{N}$,

$$A\|P_{J_r(t)}(\alpha^k)\|_{\ell_r^2}^2 \leqslant \langle R_r(t)(P_{J_r(t)}(\alpha^k)), P_{J_r(t)}(\alpha^k)\rangle_{\ell_r^2} \leqslant B\|P_{J_r(t)}(\alpha^k)\|_{\ell_r^2}^2. \tag{6.4.14}$$

Indeed, suppose (6.4.14) were false. Then, there are a measurable set $Q \subset \mathbb{T}^n$ with $m(Q) \neq 0$, $k_0 \in \mathbb{N}$ and $\varepsilon > 0$ so that at least one of the following two inequalities holds:

$$\begin{aligned}&\langle R_r(t)(P_{J_r(t)}(\alpha^{k_0})), P_{J_r(t)}(\alpha^{k_0})\rangle_{\ell_r^2} > (B+\varepsilon)\|P_{J_r(t)}(\alpha^{k_0})\|_{\ell_r^2} \quad \text{for } t \in Q,\\ &\langle R_r(t)(P_{J_r(t)}(\alpha^{k_0})), P_{J_r(t)}(\alpha^{k_0})\rangle_{\ell_r^2} < (A-\varepsilon)\|P_{J_r(t)}(\alpha^{k_0})\|_{\ell_r^2} \quad \text{for } t \in Q.\end{aligned}$$

Taking $f \in V_r$ so that $\mathscr{T}_r f(t) = \mathbf{1}_Q(t)P_{J_r(t)}(\alpha^{k_0})$ contradicts (6.4.13). Hence, (6.4.14) holds.

For the opposite implication, let $\sigma(R_r(t)) \subseteq [A,B]$ for a.e. $t \in \mathbb{T}^n$, i.e.

$$A\|\mathscr{T}_r f(t)\|_{\ell_r^2}^2 \leqslant \big\langle R_r(t)(\mathscr{T}_r f(t)), \mathscr{T}_r f(t)\big\rangle_{\ell_r^2} \leqslant B\|\mathscr{T}_r f(t)\|_{\ell_r^2}^2 \quad \text{for a.e. } t \in \mathbb{T}^n,\ f \in V_r.$$

Integrating over $\mathbb{T}^n$ gives (6.4.13).

(3) By (2), the operator $R_r(t)$ is unitary for a.e. $t \in \mathbb{T}^n$, i.e.

$$\sigma(R_r(t)R_r^*(t)) = \sigma(R_r^*(t)R_r(t)) = \{1\} \quad \text{for a.e. } t \in \mathbb{T}^n$$

if and only if $\sigma(L_rL_r^*) = \sigma(L_r^*L_r) = \{1\}$, i.e. L_r is a unitary operator. □

Using the shift-preserving operator L_r, properties of the dimension function are obtained and they are given in the following propositions.

Proposition 6.4.1 ([7])**.** *Assume that $V_r \subset H^r$ is a SI space, $L_r : V_r \to H^r$ is a shift-preserving operator and let $V_r^\circ = \overline{L_r(V_r)}$. Then, $\dim_{V_r^\circ}(t) \leqslant \dim_{V_r}(t)$ for a.e. $t \in \mathbb{T}^n$.*

Proof. By Theorem 6.3.1, $V_r = S_r(\mathscr{A}_{r,I})$. Using the theorems 6.1.1 and 6.4.2, the range function J_r° of $V_r^\circ = S_r\big(\{L_r f : f \in \mathscr{A}_{r,I}\}\big)$ satisfies

$$\begin{aligned} J_r^\circ(t) &= \overline{\operatorname{span}}\,\big\{\mathscr{T}_r f^\circ(t) : f^\circ \in \{L_r f : f \in \mathscr{A}_{r,I}\}\big\} = \overline{\operatorname{span}}\,\big\{(\mathscr{T}_r L_r)f(t) : f \in \mathscr{A}_{r,I}\big\} \\ &= \overline{\operatorname{span}}\,\big\{R_r(t)(\mathscr{T}_r f(t)) : f \in \mathscr{A}_{r,I}\big\} = \overline{R_r(t)(J_r(t))} \quad \text{for a.e. } t \in \mathbb{T}^n. \end{aligned}$$

Hence, $\dim J_r^\circ(t) \leqslant \dim J_r(t)$ for a.e. $t \in \mathbb{T}^n$. □

Proposition 6.4.2 ([7])**.** *Assume that $V_r, V_r^\circ \subset H^r$ are SI spaces. Then,*

$$\dim_{V_r}(t) = \dim_{V_r^\circ}(t) \quad \text{for a.e. } t \in \mathbb{T}^n$$

if and only if there is a shift-preserving operator $L_r : V_r \to V_r^\circ$ which is an isomorphism (or isometry).

Proof. Let $L_r : V_r \to V_r^\circ$ be a shift-preserving isomorphism. Applying Proposition 6.4.1 to $L_r : V_r \to V_r^\circ$ and $L_r^{-1} : V_r^\circ \to V_r$ gives $\dim_{V_r}(t) = \dim_{V_r^\circ}(t)$ for a.e. $t \in \mathbb{T}^n$.

On the contrary, let $V_r, V_r^\circ \subset H^r$ be SI spaces so that $\dim_{V_r}(t) = \dim_{V_r^\circ}(t)$ for a.e. $t \in \mathbb{T}^n$. According to Theorem 6.3.1,

$$V_r = \bigoplus_{k\in\mathbb{N}} S_r(f_k), \qquad V_r^\circ = \bigoplus_{k\in\mathbb{N}} S_r(f_k^\circ),$$

where f_k and f_k° are tight frame generators of $S_r(f_k)$ and $S_r(f_k^\circ)$, respectively, and $\sigma_{S_r(f_k)} = \sigma_{S_r(f_k^\circ)}$, $k \in \mathbb{N}$. Define operators $L_{r,k} : S_r(f_k) \to S_r(f_k^\circ)$ by $L_{r,k}(T_q f_k) = T_q f_k^\circ$, $k \in \mathbb{N}$. Then, for every sequence $(\alpha_q)_{q\in\mathbb{Z}^n} \in \ell_r^2$ with a finite number of non-zero elements, follows

$$\begin{aligned} \Big\|\sum_{q\in\mathbb{Z}^n} \alpha_q T_q f_k\Big\|_{H^r}^2 &= \Big\|\sum_{q\in\mathbb{Z}^n} \alpha_q M_{-q}\mathscr{T}_r f_k\Big\|_{\mathscr{H}^r}^2 = \int_{\mathbb{T}^n} \Big\|\sum_{q\in\mathbb{Z}^n} \alpha_q M_{-q}\mathscr{T}_r f_k(t)\Big\|_{\ell_r^2}^2 \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \Big|\sum_{q\in\mathbb{Z}^n} \alpha_q\, \mathrm{e}^{-2\pi i\langle t,q\rangle}\Big|^2 \|\mathscr{T}_r f_k(t)\|_{\ell_r^2}^2 \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \Big|\sum_{q\in\mathbb{Z}^n} \alpha_q\, \mathrm{e}^{-2\pi i\langle t,q\rangle}\Big|^2 \|\mathscr{T}_r f_k^\circ(t)\|_{\ell_r^2}^2 \,\mathrm{d}t \\ &= \Big\|\sum_{q\in\mathbb{Z}^n} \alpha_q T_q f_k^\circ\Big\|_{H^r}^2 = \Big\|L_{r,k}\Big(\sum_{q\in\mathbb{Z}^n} \alpha_q T_q f_k\Big)\Big\|_{H^r}^2, \end{aligned}$$

since $\dim_{V_r}(t) = \dim_{V_r^\circ}(t)$ for a.e. $t \in \mathbb{T}^n$ and thus $\|\mathscr{T}_r f_k(t)\|_{\ell_r^2} = \|\mathscr{T}_r f_k^\circ(t)\|_{\ell_r^2}^2$ for a.e. $t \in \mathbb{T}^n$ (see Theorem 6.3.1). Therefore, $L_{r,k}$ is a shift-preserving isometry. Set $L_r = \bigoplus_{k\in\mathbb{N}} L_{r,k}$. Then, the operator L_r has the desired properties. □

6.5 The frame operator and dual frame

In this section, a frame operator $F_{o,r} : S_r(\mathscr{A}_{r,I}) \to S_r(\mathscr{A}_{r,I})$ which is shift-preserving is defined, because such an operator has the range operator by Theorem 6.4.2 (1). Moreover, it will be shown that this range operator is equal to the restriction of the corresponding dual Gramian to J_r.

In order to define the operator $F_{o,r}$, it is necessary to first introduce the operator K_r and its adjoint operator K_r^*.

Definition 6.5.1 ([7]). *Let $E_r(\mathscr{A}_{r,I}) = \{T_q f_k : f_k \in \mathscr{A}_{r,I},\ k \in I,\ q \in \mathbb{Z}^n\}$ be a Bessel family of $S_r(\mathscr{A}_{r,I})$. The operator $K_r : S_r(\mathscr{A}_{r,I}) \to \ell_r^2(\mathbb{Z}^n \times I)$ is defined by*

$$K_r f = \left(\frac{\langle f, T_q f_k\rangle_{H^r}}{\mu_r(q+k)} \right)_{(q,k)\in\mathbb{Z}^n\times I}, \quad f \in S_r(\mathscr{A}_{r,I}).$$

Note, the condition that $E_r(\mathscr{A}_{r,I})$ is a Bessel family of $S_r(\mathscr{A}_{r,I})$ ensures that the operator K_r is well defined (see the conclusions given before Definition 6.2.1).

Lemma 6.5.1 ([7]). *The adjoint operator $K_r^* : \ell_r^2(\mathbb{Z}^n \times I) \to S_r(\mathscr{A}_{r,I})$ of K_r is given by*

$$K_r^* \alpha = \sum_{(q,k)\in\mathbb{Z}^n\times I} \alpha_{q,k} T_q f_k \mu_r(q+k),$$

where $\alpha = (\alpha_{q,k})_{(q,k)\in\mathbb{Z}^n\times I} \in \ell_r^2(\mathbb{Z}^n \times I)$ and $f_k \in \mathscr{A}_{r,I}$, $k \in I$.

Proof. Let $f \in S_r(\mathscr{A}_{r,I})$ and $\alpha \in \ell_r^2(\mathbb{Z}^n \times I)$. Then,

$$\begin{aligned}\langle K_r^*\alpha, f\rangle_{H^r} &= \langle \alpha, K_r f\rangle_{\ell_r^2(\mathbb{Z}^n\times I)} = \sum_{(q,k)\in\mathbb{Z}^n\times I} \alpha_{q,k} \overline{\langle f, T_q f_k\rangle_{H^r}} \mu_r(q+k) \\ &= \sum_{(q,k)\in\mathbb{Z}^n\times I} \alpha_{q,k} \langle T_q f_k, f\rangle_{H^r} \mu_r(q+k) = \left\langle \sum_{(q,k)\in\mathbb{Z}^n\times I} \alpha_{q,k} T_q f_k \mu_r(q+k), f \right\rangle_{H^r}.\end{aligned}$$

Hence, the assertion holds. □

Definition 6.5.2 ([7]). *The operator $F_{o,r} : S_r(\mathscr{A}_{r,I}) \to S_r(\mathscr{A}_{r,I})$ is defined by $F_{o,r} = K_r^* K_r$.*

Note that, $E_r(\mathscr{A}_{r,I})$ is a frame with frame bounds A and B if and only if

$$A\|f\|_{H^r}^2 \leqslant \langle F_{o,r} f, f\rangle_{H^r} \leqslant B\|f\|_{H^r}^2 \quad \text{for every } f \in S_r(\mathscr{A}_{r,I})$$

if and only if $\sigma(F_{o,r}) \subseteq [A, B]$, since

$$\langle F_{o,r} f, f\rangle_{H^r} = \langle K_r f, K_r f\rangle_{\ell_r^2(\mathbb{Z}^n\times I)} = \|K_r f\|_{\ell_r^2(\mathbb{Z}^n\times I)}^2 = \sum_{(q,k)\in\mathbb{Z}^n\times I} \left|\langle f, T_q f_k\rangle_{H^r}\right|^2.$$

Thus, it is not difficult to check that the following statement holds.

Theorem 6.5.1 ([7]). *The operator $F_{o,r} : S_r(\mathscr{A}_{r,I}) \to S_r(\mathscr{A}_{r,I})$ is a frame operator and*

$$F_{o,r}f = \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle f, T_q f_k\rangle_{H^r} T_q f_k, \quad f \in S_r(\mathscr{A}_{r,I}),$$

with the unconditional convergence in H^r.

Theorem 6.5.2 ([7]). *Assume that J_r is a range function of $V_r = S_r(\mathscr{A}_{r,I})$ and $E_r(\mathscr{A}_{r,I})$ is a Bessel family of V_r. Then, the operator $F_{o,r}$ is shift-preserving with the range operator*

$$R_r(t) = G_r^*(t) \restriction_{J_r(t)},$$

where $G_r^*(t)$ is the dual Gramian of $\{\mathscr{T}_r f_k(t) : f_k \in \mathscr{A}_{r,I},\ k \in I\}$ for a.e. $t \in \mathbb{T}^n$.

Proof. Since for every $p \in \mathbb{Z}^n$

$$\begin{aligned} F_{o,r}T_p f &= \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle T_p f, T_q f_k\rangle_{H^r} T_q f_k = \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle f, T_{q-p} f_k\rangle_{H^r} T_q f_k \\ &= \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle f, T_q f_k\rangle_{H^r} T_{q+p} f_k, \quad f \in S_r(\mathscr{A}_{r,I}), \end{aligned}$$

it follows that $F_{o,r}T_p = T_p F_{o,r}$, i.e. $F_{o,r}$ is a shift-preserving operator (obviously $F_{o,r}$ is bounded and linear). Thus, by Theorem 6.4.2,

$$\begin{aligned} \|K_r f\|^2_{\ell^2_r} &= \langle K_r f, K_r f\rangle_{\ell^2_r(\mathbb{Z}^n\times I)} = \langle F_{o,r} f, f\rangle_{H^r} = \langle (\mathscr{T}_r F_{o,r}) f, \mathscr{T}_r f\rangle_{\mathscr{H}^r} \\ &= \int_{\mathbb{T}^n} \big\langle \widetilde{R_r}(t)\big(\mathscr{T}_r f(t)\big), \mathscr{T}_r f(t)\big\rangle_{\ell^2_r} \,\mathrm{d}t, \quad f \in S_r(\mathscr{A}_{r,I}), \end{aligned} \tag{6.5.1}$$

where $\widetilde{R_r}$ is the range operator for $F_{o,r}$. Using Lemma 6.2.1 (1),

$$\begin{aligned} \|Kf\|^2_{\ell^2_r} &= \left\| \left(\frac{\langle f, T_q f_k\rangle_{H^r}}{\mu_r(q+k)} \right)_{(q,k)\in\mathbb{Z}^n\times I} \right\|^2_{\ell^2_r} \\ &= \sum_{(q,k)\in\mathbb{Z}^n\times I} \big|\langle f, T_q f_k\rangle_{H^r}\big|^2 \\ &= \sum_{k\in I} \int_{\mathbb{T}^n} \Big|\big\langle \mathscr{T}_r f(t), \mathscr{T}_r f_k(t)\big\rangle_{\ell^2_r}\Big|^2 \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \sum_{k\in I} \big\langle \mathscr{T}_r f(t), \mathscr{T}_r f_k(t)\big\rangle_{\ell^2_r} \overline{\big\langle \mathscr{T}_r f(t), \mathscr{T}_r f_k(t)\big\rangle_{\ell^2_r}} \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \Big\langle \big(\langle \mathscr{T}_r f(t), \mathscr{T}_r f_k(t)\rangle_{\ell^2_r}\big)_{k\in I}, \big(\langle \mathscr{T}_r f(t), \mathscr{T}_r f_k(t)\rangle_{\ell^2_r}\big)_{k\in I}\Big\rangle_{\ell^2} \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \big\langle D_r^* \mathscr{T}_r f(t), D_r^* \mathscr{T}_r f(t)\big\rangle_{\ell^2} \,\mathrm{d}t \\ &= \int_{\mathbb{T}^n} \big\langle D_r D_r^* \mathscr{T}_r f(t), \mathscr{T}_r f(t)\big\rangle_{\ell^2_r} \,\mathrm{d}t \qquad (6.5.2) \\ &= \int_{\mathbb{T}^n} \big\langle G_r^*(t) \mathscr{T}_r f(t), \mathscr{T}_r f(t)\big\rangle_{\ell^2_r} \,\mathrm{d}t, \quad f \in S_r(\mathscr{A}_{r,I}). \qquad (6.5.3) \end{aligned}$$

Now, combining (6.5.1) and (6.5.2) yields

$$\int_{\mathbb{T}^n} \big\langle \big(\widetilde{R_r}(t) - G_r^*(t) \restriction_{J_r(t)}\big)\big(\mathscr{T}_r f(t)\big), \mathscr{T}_r f(t)\big\rangle_{\ell^2_r} \,\mathrm{d}t = 0, \quad f \in S_r(\mathscr{A}_{r,I}).$$

Hence, $\widetilde{R}_r(t) = G_r^*(t) \restriction_{J_r(t)} = R_r(t)$ for a.e. $t \in \mathbb{T}^n$. $\square$

Finally, for a frame with given frame bounds A and B a dual frame is determined with frame bounds B^{-1} and A^{-1}.

Theorem 6.5.3 ([7])**.** *Let $E_r(\mathscr{A}_{r,I})$ be a frame of $V_r = S_r(\mathscr{A}_{r,I})$ with frame bounds A, B and let $\mathscr{B}_{r,I} = \{f_k^* : f_k^* = F_{o,r}^{-1} f_k,\ f_k \in \mathscr{A}_{r,I},\ k \in I\}$. Then, $E_r(\mathscr{B}_{r,I})$ is the dual frame of $E_r(\mathscr{A}_{r,I})$ with frame bounds B^{-1}, A^{-1}, and*

$$\mathscr{T}_r f_k^*(t) = R_r^{-1}(t)\big(\mathscr{T}_r f_k(t)\big) \quad \text{for a.e. } t \in \mathbb{T}^n,\ k \in I. \tag{6.5.4}$$

Proof. Using Theorem 6.5.1, for every $f \in V_r$,

$$\begin{aligned}
\sum_{(q,k)\in\mathbb{Z}^n\times I} \left|\langle f, F_{o,r}^{-1} T_q f_k\rangle_{H^r}\right|^2 &= \sum_{(q,k)\in\mathbb{Z}^n\times I} \left|\langle F_{o,r}^{-1} f, T_q f_k\rangle_{H^r}\right|^2 \\
&= \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle F_{o,r}^{-1} f, T_q f_k\rangle_{H^r} \overline{\langle F_{o,r}^{-1} f, T_q f_k\rangle}_{H^r} \\
&= \sum_{(q,k)\in\mathbb{Z}^n\times I} \big\langle \langle F_{o,r}^{-1} f, T_q f_k\rangle_{H^r} T_q f_k, F_{o,r}^{-1} f\big\rangle_{H^r} \\
&= \Big\langle \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle F_{o,r}^{-1} f, T_q f_k\rangle_{H^r} T_q f_k, F_{o,r}^{-1} f\Big\rangle_{H^r} \\
&= \big\langle F_{o,r}(F_{o,r}^{-1} f), F_{o,r}^{-1} f\big\rangle_{H^r} \\
&= \langle F_{o,r}^{-1} f, f\rangle_{H^r}.
\end{aligned}$$

Theorem 5.2.1 (3) gives

$$B^{-1}\|f\|_{H^r} \leqslant \sum_{(q,k)\in\mathbb{Z}^n\times I} \left|\langle f, F_{o,r}^{-1} T_q f_k\rangle_{H^r}\right|^2 = \langle F_{o,r}^{-1} f, f\rangle_{H^r} \leqslant A^{-1}\|f\|_{H^r}. \tag{6.5.5}$$

Therefore, $\{F_{o,r}^{-1} T_q f_k : f_k \in \mathscr{A}_{r,I},\ q \in \mathbb{Z}^n,\ k \in I\}$ is a dual frame for $E_r(\mathscr{A}_{r,I})$ with frame bounds B^{-1}, A^{-1}. Further, for every $f^\circ = F_{o,r} f \in H^r$,

$$F_{o,r}^{-1} T_q f^\circ = F_{o,r}^{-1} T_q F_{o,r} f = F_{o,r}^{-1} F_{o,r} T_q f = T_q f = T_q F_{o,r}^{-1} f^\circ, \quad q \in \mathbb{Z}^n,$$

since the operator $F_{o,r}$ is shift-preserving, by Theorem 6.5.2. Hence, $F_{o,r}^{-1}$ is also a shift-preserving operator. Now, by (6.5.5), it follows that $E_r(\mathscr{B}_{r,I})$ is a dual frame for $E_r(\mathscr{A}_{r,I})$ with frame bounds B^{-1}, A^{-1}, and hold

$$f = \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle f, T_q f_k^*\rangle_{H^r} T_q f_k = \sum_{(q,k)\in\mathbb{Z}^n\times I} \langle f, T_q f_k\rangle_{H^r} T_q f_k^*, \quad f \in V_r,$$

with the unconditional convergence in H^r.

Finally, (6.5.4) follows from (6.4.6) and Theorem 6.5.2. □

Remark 6.5.1. *Let $E_r(\mathscr{A}_{r,I})$ be a Riesz family for $V_r = S_r(\mathscr{A}_{r,I})$ with frame bounds A, B. Then, the dual $E_r(\mathscr{B}_{r,I})$ is also a Riesz family with bounds B^{-1}, A^{-1}. Moreover,*

$$\langle T_q f_k, T_p f_j^*\rangle_{H^r} = \delta_{q,p}\delta_{k,j}, \quad q, p \in \mathbb{Z}^n,\ k, j \in I,$$

where $\delta_{k,j}$ is Kronecker's delta function.

6.6 The structure theorem and connection with another approach

Note that V_r is a separable Hilbert space, since V_r is a closed subspace of H^r, $r \in \mathbb{R}$. The space $L^2_{pe} = L^2_{pe}(\mathbb{R}^n)$ of L^2-periodic functions is defined by

$$L^2_{pe} = \Big\{ f^\star : f^\star(\cdot) = \sum_{q\in\mathbb{Z}^n} \alpha_q \, \mathrm{e}^{-2\pi i \langle \cdot, q\rangle}, \ (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell^2 \Big\}.$$

Now, using the Fourier transform, a characterization of elements of V_r is obtained.

Theorem 6.6.1 (**The structure theorem,** [6]). *Let $V_r = S_r(\mathscr{A}_{r,I})$, $E_r(\mathscr{A}_{r,I})$ be a frame of V_r and let $E_r(\mathscr{B}_{r,I})$ be its dual frame, where $\mathscr{B}_{r,I} = \{f^*_k : f^*_k = F^{-1}_{o,r} f_k,\ f_k \in \mathscr{A}_{r,I},\ k \in I\}$. Then, the Fourier transform of V_r, i.e. $\mathscr{F}[V_r]$ is the set of the Fourier transforms of elements from $\mathscr{D}'_{L^2}$ so that*

$$\mathscr{F}[f] = \sum_{k\in I} \mathscr{F}[f_k] f^\star_k,$$

where $\mathscr{F}[f_k] \in L^2_r$, $k \in I$, and $f^\star_k \in L^2_{pe}$ have the expansions

$$f^\star_k = \sum_{q\in\mathbb{Z}^n} \alpha_{q,k} \, \mathrm{e}^{-2\pi i\langle \cdot, q\rangle}, \quad (\alpha_{q,k})_{(q,k)\in\mathbb{Z}^n\times I} \in \ell^2(\mathbb{Z}^n \times I),$$

with

$$\alpha_{q,k} = \int_{\mathbb{R}^n} \mathscr{F}[f](x) \, \mathrm{e}^{2\pi i\langle q,x\rangle} \, \overline{\mathscr{F}[f^*_k](x)} \mu^2_r(x) \, \mathrm{d}x, \quad (q,k) \in \mathbb{Z}^n \times I. \tag{6.6.1}$$

Proof. By Theorem 6.5.3, the existence of a dual frame is insured. Thus, let $E_r(\mathscr{A}_{r,I})$ and $E_r(\mathscr{B}_{r,I})$ be a frame and its dual frame of $V_r = S_r(\mathscr{A}_{r,I})$, respectively. Then, using Lemma 5.3.1, it follows that

$$f = \sum_{k\in I}\sum_{q\in\mathbb{Z}^n} \langle f, T_q f_k\rangle_{H^r} T_q f^*_k = \sum_{k\in I}\sum_{q\in\mathbb{Z}^n} \langle f, T_q f^*_k\rangle_{H^r} T_q f_k = \sum_{k\in I}\sum_{q\in\mathbb{Z}^n} \alpha_{q,k} T_q f_k$$

for every $f \in V_r$, where

$$\alpha_{q,k} = \langle f, T_q f^*_k\rangle_{H^r} = \int_{\mathbb{R}^n} \mathscr{F}[f](x) \, \mathrm{e}^{2\pi i\langle x,q\rangle} \, \overline{\mathscr{F}[f^*_k](x)} \mu^2_r(x) \, \mathrm{d}x$$

for all $(q,k) \in \mathbb{Z}^n \times I$. Moreover, by (5.2.1),

$$A\|f\|^2_{H^r} \leqslant \sum_{k\in I}\sum_{q\in\mathbb{Z}^n} |\alpha_{q,k}|^2 \leqslant B\|f\|^2_{H^r}, \quad f \in V_r,$$

since $E_r(\mathscr{B}_{r,I})$ is a frame. Therefore, $(\alpha_{q,k})_{(q,k)\in\mathbb{Z}^n\times I} \in \ell^2(\mathbb{Z}^n \times I)$. □

In the next assertions, different approaches to SI spaces are connected. First, the SI space $V_0 = V$ is related to the SI space from [15] (see (1.0.2) for $r = 0$) by the following statement.

Theorem 6.6.2 ([6]). *Let $V = S(\mathscr{A}_m)$, where $\mathscr{A}_m = \{f_k : k = 1, \dots, m\} \subset L^2 \cap \mathscr{L}^\infty$. If $\mathscr{V}_0$ is closed in L^2, then $\mathscr{V}_0 = V$.*

Proof. According to Theorem 1.0.5, $\mathscr{V}_0$ is closed in L^2 if and only if $E(\mathscr{A}_m)$ is a frame of $\mathscr{V}_0$. Moreover, by the definition, $V = S(\mathscr{A}_m)$ is closed in L^2. Therefore, the same frame determines $\mathscr{V}_0$ and V. Thus, $\mathscr{V}_0 = V$. $\square$

Shift-invariant spaces V_r, $r > 0$, are connected to weighted SI spaces from [64], i.e. the spaces

$$\mathscr{V}_r = \left\{ f : f = \sum_{k=1}^{m} \sum_{q \in \mathbb{Z}^n} \alpha_{q,k} T_q f_k,\ (\alpha_{q,k})_{q\in\mathbb{Z}^n} \in \ell_r^2,\ f_k \in \mathscr{L}^\infty \cap L_r^2,\ k = 1, \dots, m \right\}, \quad (6.6.2)$$

by imposing additional conditions for generators.

Theorem 6.6.3 ([6]). *Let $r > 0$ and $V_r = S_r(\mathscr{A}_{r,m})$, where $\mathscr{A}_{r,m} = \{f_k \in H^r : f_k \in L_r^2 \cap \mathscr{L}^\infty,\ k = 1, \dots, m\}$.*

(1) *If $\mathscr{V}_r$ and $\mathscr{F}[\mathscr{V}_r]$ are closed in L_r^2, then $\mathscr{V}_r \subset H^r$ and $\mathscr{V}_r = V_r$, i.e. every $f \in V_r$ has the expansion as in (6.6.2).*

(2) *If $r > \frac{1}{2}$ and $\mathscr{V}_r$ is closed in L_r^2, then $\mathscr{F}[\mathscr{V}_r]$ is closed in L_r^2 and both assertions in (1) hold.*

Proof. (1) On the one hand, by Theorem 1.0.6, $\mathscr{V}_r$ is closed in L_r^2 if and only if $E_r(\mathscr{A}_{r,m})$ is a frame of $\mathscr{V}_r$. On the other hand, by Lemma 3.5.2, $\mathscr{F}[H^r] = L_r^2$ and thus $\mathscr{F}^{-1}[\mathscr{F}\mathscr{V}_r] = \mathscr{V}_r$ is a closed subspace of H^r, because $\mathscr{F}[\mathscr{V}_r]$ is closed in L_r^2 and $\mathscr{F}$ is an isomorphism (Theorem 3.4.2). Therefore, since $V_r = S_r(\mathscr{A}_{r,m})$ is a closed subspace of H^r, it implies that $\mathscr{V}_r = V_r$. Hence, $f \in V_r$ has the expansion as in (6.6.2).

(2) If $f \in \mathscr{V}_r$, then

$$f = \sum_{k=1}^{m} \sum_{q \in \mathbb{Z}^n} \alpha_{q,k} T_q f_k \quad \text{and} \quad \widehat{f} = \sum_{k=1}^{m} \widehat{f}_k \sum_{q \in \mathbb{Z}^n} \alpha_{q,k}\, \mathrm{e}^{-2\pi i \langle \cdot, q \rangle}.$$

Let

$$\widehat{f}_N = \sum_{k=1}^{m} \widehat{f}_k \sum_{|q|>N} \alpha_{q,k}\, \mathrm{e}^{-2\pi i \langle \cdot, q \rangle}.$$

In order to prove that $\widehat{f} \in L_r^2$ it is enough to prove

$$\int_{\mathbb{R}^n} \widehat{f}_N(t) \overline{\widehat{f}_N(t)} \mu_r^2(t)\, \mathrm{d}t \to 0, \quad N \to +\infty.$$

Since

$$\begin{aligned} \widehat{f}_N(t)\overline{\widehat{f}_N(t)} &= \sum_{k_1,k_2=1}^{m} \widehat{f}_{k_1}(t)\overline{\widehat{f}_{k_2}(t)} \sum_{|q|>N} \alpha_{q,k_1}\, \mathrm{e}^{-2\pi i \langle t, q \rangle} \sum_{|q|>N} \overline{\alpha_{q,k_2}}\, \mathrm{e}^{2\pi i \langle t, q \rangle} \\ &= \sum_{k_1,k_2=1}^{m} \widehat{f}_{k_1}(t)\overline{\widehat{f}_{k_2}(t)} I_{k_1,k_2,N}, \end{aligned}$$

and

$$\widehat{f}_{k_1}(t)\overline{\widehat{f}_{k_2}(t)} \mu_r^2(t) \in L^2,$$

it is enough to prove

$$|I_{k_1,k_2,N}| \leqslant \sup_{t\in\mathbb{R}^n} \Big| \sum_{|q|>N} \alpha_{q,k_1} \,\mathrm{e}^{-2\pi i\langle t,q\rangle} \sum_{|q|>N} \overline{\alpha_{q,k_2}} \,\mathrm{e}^{2\pi i\langle t,q\rangle} \Big| \to 0, \quad N\to+\infty.$$

Since $(\alpha_{q,k})_{q\in\mathbb{Z}^n} \in \ell^2_r$, $k=1,\dots,m$, using the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
|I_{k_1,k_2,N}| &\leqslant \sum_{|q|>N} |\alpha_{q,k_1}| \sum_{|q|>N} |\alpha_{q,k_2}| \\
&= \sum_{|q|>N} |\alpha_{q,k_1}|\mu_r(q)\mu_{-r}(q) \sum_{|q|>N} |\alpha_{q,k_2}|\mu_r(q)\mu_{-r}(q) \\
&\leqslant \sum_{|q|>N} |\alpha_{q,k_1}|^2\mu_r^2(q) \sum_{|q|>N} \mu_{-r}^2(q) \sum_{|q|>N} |\alpha_{q,k_2}|^2\mu_r^2(q) \sum_{|q|>N} \mu_{-r}^2(q) \to 0, \quad N\to+\infty.
\end{aligned}$$

Therefore, the statement holds. □

Regarding duality, the following statement holds.

Theorem 6.6.4 ([6])**.** *Let $r>0$ and $V_r = S_r(\mathscr{A}_{r,m})$, where $\mathscr{A}_{r,m} = \{f_k \in H^r : f_k \in L^2_r \cap \mathscr{L}^\infty,\ k=1,\dots,m\}$. If the conditions of assertion* (1) *or* (2) *in Theorem* 6.6.3 *hold, then*

(1) *$\mathscr{V}_r' = \mathscr{V}_{-r}$, where $\mathscr{V}_{-r}$ is the space of series of the form*

$$f^* = \sum_{k=1}^m \sum_{q\in\mathbb{Z}^n} \beta_{q,k} T_q f_k, \quad \sum_{k=1}^m \sum_{q\in\mathbb{Z}^n} |\beta_{q,k}|^2 \mu_{-r}^2(q) < +\infty,$$

with the dual pairing

$$\langle f^*, f\rangle_{\mathscr{V}_r',\mathscr{V}_r} = \sum_{k=1}^m \sum_{q\in\mathbb{Z}^n} \beta_{q,k}\alpha_{q,k}, \quad f\in\mathscr{V}_r,$$

(2) *$\mathscr{V}_{-r} = V_{-r}$.*

Proof. Obviously, the statement (1) holds. Since elements of the form

$$\sum_{k=1}^m \sum_{q\in\mathbb{Z}^n} \beta_{q,k} T_q f_k$$

are dense in the spaces $\mathscr{V}_{-r}$ and V_{-r}, for $r>0$, it implies that the statement (2) holds. □

For the equality between the spaces of intersections of the observed spaces, it is necessary to consider generators from the Schwartz space $\mathscr{S}$.

Theorem 6.6.5 ([6])**.** *Let $f_k \in \mathscr{S}$, $k=1,\dots,m$. Then,*

$$\bigcap_{r\geqslant 0} \mathscr{V}_r = \bigcap_{r\geqslant 0} V_r,$$

and its elements can be represented in the form (6.6.2) *with*

$$\sup_{q\in\mathbb{Z}^n} |\alpha_{q,k}||q|^r < +\infty \quad \text{for all } r\geqslant 0,\ k=1,\dots,m.$$

A direct consequence of the theorems 6.6.3 and 6.6.4 is the following assertion.

Corollary 6.6.1 ([6]). *Let $f_k \in \mathscr{S}$, $k = 1, \ldots, m$. Then:*

(1)

$$\mathscr{F}\Big[\bigcap_{r\geqslant 0} \mathscr{V}_r\Big] = \Big\{ \sum_{k=1}^{m} \widehat{f}_k w_k : w_k \in \mathscr{P},\ k = 1, \ldots, m \Big\},$$

(2) $V'_r = \mathscr{V}_{-r}$, $\bigcup_{r\geqslant 0} V'_r = \bigcup_{r\geqslant 0} \mathscr{V}_{-r}$ *and*

$$\mathscr{F}\Big[\bigcup_{r\leqslant 0} \mathscr{V}_r\Big] = \Big\{ \sum_{k=1}^{m} \widehat{f}_k v_k : v_k \in \mathscr{P}',\ k = 1, \ldots, m \Big\}.$$

6.7 Spectral analysis of the range operator

This section is devoted to the range operator. Note, in the continuation with **1**, i.e. $1(t)$, will be denoted a unit mapping or a unit matrix (it will be clear from the context whether it is a matrix or a mapping).

Theorem 6.7.1 ([4]). *Let $\Xi \subset \mathbb{R}^n$ be a measurable set. If $[A(t)]_{m\times m}$ is a matrix of measurable functions defined on Ξ, then there are m measurable functions $\lambda_k : \Xi \to \mathbb{C}$, $k = 1, \ldots, m$, so that $\lambda_1(t), \ldots, \lambda_m(t)$ are eigenvalues of matrix $[A(t)]_{m\times m}$ for a.e. $t \in \Xi$.*

Theorem 6.7.2 ([7]). *Let $\Xi \subset \mathbb{T}^n$ be a measurable set. Assume that $V_r \subset H^r$ is a SI space with the range function J_r, and $R_r : J_r \to J_r$ is the corresponding range operator for a shift-preserving operator $L_r : V_r \to V_r$. If*

$$\dim_{V_r}(t) = m < +\infty \quad \text{for a.e. } t \in \Xi,$$

then there are m^2 measurable bounded functions $(R_r^{k,j})_{k,j=1}^m$ defined on Ξ so that

$$R_r(t) = \begin{bmatrix} R_r^{1,1}(t) & R_r^{1,2}(t) & \cdots & R_r^{1,m}(t) \\ R_r^{2,1}(t) & R_r^{2,2}(t) & \cdots & R_r^{2,m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ R_r^{m,1}(t) & R_r^{m,2}(t) & \cdots & R_r^{m,m}(t) \end{bmatrix}_{m\times m} \qquad \text{for a.e. } t \in \Xi.$$

Furthermore, there are m measurable functions $\lambda_{r,k} : \Xi \to \mathbb{C}$, $k = 1, \ldots, m$, such that $\lambda_{r,1}(t), \ldots, \lambda_{r,m}(t)$ are eigenvalues for $R_r(t)$ for a.e. $t \in \Xi$, counted with multiplicity.

Proof. Choose the sequence of sets $(A_m)_{m\in\mathbb{N}_0}$ and $f_k \in H^r$, $k \in \mathbb{N}$, from Proposition 6.3.1. Since the set $\{\mathscr{T}_r f_1(t), \ldots, \mathscr{T}_r f_m(t)\}$ is an orthonormal basis of $J_r(t)$ for a.e. $t \in A_m$, it follows that the range operator $R_r(t)$ has the matrix representation,

$$R_r^{k,j}(t) = \big\langle R_r(t)\mathscr{T}_r f_j(t), \mathscr{T}_r f_k(t)\big\rangle_{\ell_r^2} \quad \text{for a.e. } t \in A_m.$$

Obviously, the elements $R_r^{k,j}(t)$, $k, j = 1, \ldots, m$, for a.e. $t \in A_m$, are measurable functions. Moreover, by Theorem 6.4.2, holds $|R_r^{k,j}(t)| \leqslant \|L_r\|$ for a.e. $t \in A_m$ and all $k, j = 1, \ldots, m$, because L_r is bounded. Since $\dim_{V_r}(t) = m$ for a.e. $t \in \Xi$, it implies that $\Xi \subseteq A_m$ (Proposition 6.3.1). Therefore, the first part of the statement holds.

Further, let $\{e_1,\dots,e_m\}$ be the canonical basis of $\mathbb{C}^m$. Define a mapping $\rho_r(t) : J_r(t) \to \mathbb{C}^m$ by $\rho_r(t)\big(\mathscr{T}_r f_k(t)\big) = e_k$ for a.e. $t \in A_m$, $k = 1,\dots,m$. Thus, the (unique) relation between the bases is established for a.e. $t \in A_m$. Let $\lambda_r : \Xi \to \mathbb{C}$ be a measurable function. Then,

$$R_r(t) = \rho_r(t)^{-1}[R_r^{k,j}(t)]_{m\times m}\rho_r(t)$$

and $\ker\big(R_r(t) - \lambda_r(t)1(t)\big) = \ker\big(\rho_r(t)^{-1}([R_r^{k,j}(t)]_{m\times m} - \lambda_r(t)1(t))\rho_r(t)\big)$, for a.e. $t \in \Xi$. Hence, by Theorem 6.7.1, the statement follows. $\square$

In the continuation, $\mathfrak{R}_{\lambda_r}$ denotes the set of eigenvalues of the bounded measurable range operator R_r.

Definition 6.7.1 ([7]). *Let $V_r \subset H^r$ be a FSI space. The smallest $m \in \mathbb{N}$ so that $V_r = S_r(f_1,\dots,f_m)$ is called the length of V_r and it is denoted by $\mathrm{D}(V_r)$.*

Note, the equivalent definition is $\mathrm{D}(V_r) = \operatorname{ess\,sup}_{t\in\mathbb{T}^n} \dim_{V_r}(t)$.

Theorem 6.7.3 ([7]). *Assume that $V_r \subset H^r$ is a SI space with the range function J_r, and $R_r : J_r \to J_r$ is the corresponding bounded measurable range operator. Then, there are functions $\lambda_{r,k} \in L^\infty(\mathbb{T}^n)$, $k \in \mathbb{N}$, so that*

(1) *$\lambda_{r,k}(t) \neq \lambda_{r,j}(t)$, $k \neq j$, for a.e. $t \in \mathbb{T}^n$, and*

(2) *if $A_{m,d} = \{t \in A_m : \mathrm{card}(\mathfrak{R}_\lambda(t)) = d\}$, where $(A_m)_{m\in\mathbb{N}_0}$ are sets from Proposition 6.3.1, then $\mathfrak{R}_{\lambda_r}(t) = \{\lambda_{r,1}(t),\dots,\lambda_{r,d}(t)\}$ for a.e. $t \in A_{m,d}$, $d \leqslant m$.*

Proof. First, by Proposition 6.3.1, it follows that $A_k \cap A_j = \emptyset$, $k \neq j$, and $\bigcup_{m\in\mathbb{N}} A_m = \sigma_{V_r}$. Using Theorem 6.7.2, it follows that for every $m \in \mathbb{N}$ there are m measurable functions $\lambda_{r,m}^k : A_m \to \mathbb{C}$, $k = 1,\dots,m$, such that $\lambda_{r,m}^1(t),\dots,\lambda_{r,m}^m(t)$ are eigenvalues for $R_r(t)$ for a.e. $t \in A_m$, counted with multiplicity. Fix $m \in \mathbb{N}$ and define

$$A_{m,d} = \big\{t \in A_m : \mathrm{card}\{\lambda_{r,m}^1(t),\dots,\lambda_{r,m}^m(t)\} = d\big\}, \quad d \leqslant m.$$

These sets are measurable, disjoint and $\bigcup_{d=1}^m A_{m,d} = A_m$. Now, there are measurable functions $\lambda_{r,m}^{d,1},\dots,\lambda_{r,m}^{d,d} : A_{m,d} \to \mathbb{C}$ so that $\lambda_{r,m}^{d,k}$, $k = 1,\dots,d$, are eigenvalues for $R_r(t)$ for a.e. $t \in A_{m,d}$, and $\lambda_{r,m}^{d,k}(t) \neq \lambda_{r,m}^{d,j}(t)$, $k \neq j$, for a.e. $t \in A_{m,d}$. Since R_r is bounded, it gives $|\lambda_{r,m}^{d,k}(t)| \leqslant C$, $k \leqslant d \leqslant m$, for a.e. $t \in A_{m,d}$. Define $\lambda_{r,k} : \mathbb{T}^n \to \mathbb{C}$ by

$$\lambda_{r,k}(t) = \begin{cases} \lambda_{r,m}^{d,k}(t), & t \in A_{m,d},\ k \leqslant d \leqslant m, \\ C + k, & \text{otherwise.} \end{cases}$$

Then, $\lambda_{r,k}(t) \neq \lambda_{r,j}(t)$, $k \neq j$, for a.e. $t \in \mathbb{T}^n$, and $\lambda_{r,k} \in L^\infty(\mathbb{T}^n)$, $k \in \mathbb{N}$. Moreover, $\lambda_{r,k}(t)$ is the eigenvalue for $R_r(t)$ for a.e. $t \in A_{m,d}$, since for a.e. $t \in A_{m,d}$,

$$\ker\big(R_r(t) - \lambda_{r,k}(t)1(t)\big) = \ker\big(R_r(t) - \lambda_{r,m}^{d,k}(t)1(t)\big), \quad k \leqslant d \leqslant m.$$

Otherwise $\ker\big(R_r(t) - \lambda_{r,k}(t)1(t)\big) = \{\mathbf{0}\}$, because $\lambda_{r,k}(t) = C + k$ is not the eigenvalue for $R_r(t)$. $\square$

The following remark will be used in proofs of several theorems.

Remark 6.7.1. (1) *Let $V_r \subset H^r$ be a FSI space with the range function J_r. If $m > \mathrm{D}(V_r)$, then $m(A_m) = 0$. Thus, define (for $d \in \mathbb{N}$)*

$$B_d = \bigcup_{m=d}^{+\infty} A_{m,d} \quad \text{and} \quad \kappa = \max\{d \in \mathbb{N} : m(B_d) \neq 0\}. \tag{6.7.1}$$

(2) *Define the sequence of sets* $C_k = \bigcup_{d=k}^{+\infty} B_d$, $k \in \mathbb{N}$. *Then,*

$$C_k = \{t \in \sigma_{V_r} : R_r(t) \text{ has at least } k \text{ different eigenvalues}\}, \quad k \in \mathbb{N}.$$

By the proof of Theorem 6.7.3, *it is clear that*

$$C_k = \big\{t \in \mathbb{T}^n : \ker(R_r(t) - \lambda_{r,k}(t)1(t)) \neq \{\mathbf{0}\}\big\}, \quad k \in \mathbb{N},$$

and $m(C_k) = 0$ *for* $k > \kappa$.

6.8 s-Diagonalization for shift-preserving operators

The term s-diagonalization first was introduced as the definition by A. Aguilera et al. in [4]. In this section, the definition of s-diagonalization is adapted to H^r spaces. Then, it is proved that if the shift-preserving operator L_r is normal, then it is also s-diagonalizable. Moreover, if L_r is s-diagonalizable, then it can be represented via a finite sum of products of eigenvalues and corresponding orthogonal projections. Also, the s-diagonalization of the shift-preserving operator L_r and the diagonalization of the range operator R_r are connected.

Definition 6.8.1 ([7])**.** *The operator* $\mathrm{M}_a : \mathscr{H}^r \to \mathscr{H}^r$ *defined by*

$$\mathrm{M}_a \mathscr{T}_r f(t) = \left(\frac{a(t)\widehat{g}(t+q)}{\mu_r(q)}\right)_{q\in\mathbb{Z}^n}, \quad t \in \mathbb{T}^n,$$

where $a : \mathbb{T}^n \to \mathbb{C}$ *is a measurable function and* $(\mathbf{1} - \frac{1}{4\pi^2}\Delta)^{r/2} f = g \in L^2$, *is called the multiplication operator.*

Lemma 6.8.1 ([7])**.** *The operator* M_a *is continuous if and only if* $a \in L^\infty(\mathbb{T}^n)$.

Proof. Obviously, the necessary condition holds. Let $f \in H^r$ and $a \in L^\infty(\mathbb{T}^n)$. Then,

$$\begin{aligned}\|\mathrm{M}_a \mathscr{T}_r f\|_{\mathscr{H}^r}^2 &= \int_{\mathbb{T}^n} \|\mathrm{M}_a \mathscr{T}_r f(t)\|_{\ell_r^2}^2 \,\mathrm{d}t = \int_{\mathbb{T}^n} \left\| \left(\frac{a(t)\widehat{g}(t+q)}{\mu_r(q)}\right)_{q\in\mathbb{Z}^n} \right\|_{\ell_r^2}^2 \mathrm{d}t \\ &\leqslant \|a\|_{L^\infty(\mathbb{T}^n)}^2 \|\mathscr{T}_r f\|_{\mathscr{H}^r}^2,\end{aligned}$$

i.e. M_a is a continuous operator. Hence, the assertion holds. □

Definition 6.8.2 ([7])**.** *The operators* $\widehat{\alpha} : \mathbb{T}^n \to \mathbb{C}$ *and* $\Lambda_{r,\alpha} : H^r \to H^r$ *are defined by*

$$\widehat{\alpha} = \sum_{q\in\mathbb{Z}^n} \alpha_q \,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle}, \qquad \Lambda_{r,\alpha} = \sum_{q\in\mathbb{Z}^n} \alpha_q T_q,$$

where $\alpha = (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell_r^2$. *A sequence* $\alpha = (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell_r^2$ *is said to be a sequence of bounded spectrum if* $\widehat{\alpha} \in L^\infty(\mathbb{T}^n)$.

Definition 6.8.3 ([7])**.** *An operator* $\Lambda_{r,\alpha}$, *where* $\alpha = (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell_r^2$ *is a sequence of bounded spectrum, is said to be an s-eigenvalue of operator* L_r *if*

$$V_{r,\alpha} = \{f \in H^r : L_r f = \Lambda_{r,\alpha} f\} \neq \{\mathbf{0}\}.$$

The space $V_{r,\alpha}$ *is called the s-eigenspace associated with* $\Lambda_{r,\alpha}$.

The term s-eigenvalues of operator L_r generalizes its eigenvalues as shown in the following example.

Example 6.8.1. *Let $\lambda_r \in \mathbb{C}$ be an eigenvalue for L_r, i.e.* $\ker(L_r - \lambda_r \mathbf{1}) \neq \{\mathbf{0}\}$. *Then, taking the sequence $\alpha = \lambda_r \chi_0$, $\Lambda_{r,\alpha}$ is an s-eigenvalue of L_r, where $\chi_0(q) = \delta_{0,q}$, $q \in \mathbb{Z}^n$.*

The next lemma is significant, because it shows when the operator $\Lambda_{r,\alpha}$ is well defined, bounded, and moreover that it can be represented as a composition of operators. That composition is used to determine the eigenvalue of the range operator R_r, which is important for further results.

Lemma 6.8.2 ([7])**.** *If $\alpha = (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell^2_r$, then the linear operator*

$$\Lambda_{r,\alpha} = \mathscr{T}_r^{-1} \mathrm{M}_{\widehat{\alpha}} \mathscr{T}_r : H^r \to H^r$$

is a bounded operator if and only if α is a sequence of bounded spectrum.

Proof. Let $f \in H^r$ and $(\mathbf{1} - \frac{1}{4\pi^2}\Delta)^{r/2} f = g \in L^2$. Then, using Lemma 6.1.2, it follows that

$$\begin{aligned}\mathscr{T}_r^{-1}\mathrm{M}_{\widehat{\alpha}}\mathscr{T}_r f(t) &= \mathscr{T}_r^{-1}\left(\frac{\widehat{\alpha}(t)\widehat{g}(t+p)}{\mu_r(p)}\right)_{p\in\mathbb{Z}^n} = \sum_{q\in\mathbb{Z}^n} \alpha_q \mathscr{T}_r^{-1}\,\mathrm{e}^{-2\pi i\langle t,q\rangle}\left(\frac{\widehat{g}(t+p)}{\mu_r(p)}\right)_{p\in\mathbb{Z}^n}\\ &= \sum_{q\in\mathbb{Z}^n} \alpha_q \mathscr{T}_r^{-1}\,\mathrm{e}^{-2\pi i\langle t,q\rangle}\,\mathscr{T}_r f(t) = \sum_{q\in\mathbb{Z}^n} \alpha_q T_q f(t) = \Lambda_{r,\alpha} f(t),\end{aligned}$$

i.e. $\Lambda_{r,\alpha} = \mathscr{T}_r^{-1}\mathrm{M}_{\widehat{\alpha}}\mathscr{T}_r$. Linearity follows directly from the definition of $\Lambda_{r,\alpha}$. Since

$$\begin{aligned}\|\mathscr{T}_r^{-1}\mathrm{M}_{\widehat{\alpha}}\mathscr{T}_r f\|^2_{H^r} = \|\mathrm{M}_{\widehat{\alpha}}\mathscr{T}_r f\|^2_{\mathscr{H}^r} &= \int_{\mathbb{T}^n} \left\| \left(\frac{\widehat{\alpha}(t)\widehat{g}(t+p)}{\mu_r(p)}\right)_{p\in\mathbb{Z}^n} \right\|^2_{\ell^2_r} \mathrm{d}t\\ &\leqslant \|\widehat{\alpha}\|^2_{L^\infty(\mathbb{T}^n)} \|\mathscr{T}_r f\|^2_{\mathscr{H}^r} = \|\widehat{\alpha}\|^2_{L^\infty(\mathbb{T}^n)} \|f\|^2_{H^r},\end{aligned}$$

the statement follows. □

The following lemmas are necessary for the proof of Theorem 6.8.1.

Lemma 6.8.3 ([7])**.** *Let α be a sequence of bounded spectrum. Then, for every $f \in V_{r,\alpha}$*

$$R_r(t)\big(\mathscr{T}_r f(t)\big) = \widehat{\alpha}(t)\mathscr{T}_r f(t) \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Proof. Let $f \in V_{r,\alpha}$. Using the equality (6.4.6), $L_r f = \Lambda_{r,\alpha} f$ and Lemma 6.8.2, it follows that

$$R_r(t)\big(\mathscr{T}_r f(t)\big) = \mathscr{T}_r\big(L_r f\big)(t) = \mathscr{T}_r\big(\Lambda_{r,\alpha} f\big)(t) = \mathscr{T}_r\big(\mathscr{T}_r^{-1}\mathrm{M}_{\widehat{\alpha}}\mathscr{T}_r f\big)(t) = \widehat{\alpha}(t)\mathscr{T}_r f(t),$$

for a.e. $t \in \mathbb{T}^n$. Therefore, the statement holds. □

The following auxiliary statement is a direct consequence of Proposition 2.9 [22].

Lemma 6.8.4 ([7])**.** *If $V_r \subset H^r$ is a SI space, then there is $f \in V_r$ so that*

$$\operatorname{supp} \|\mathscr{T}_r f\|_{\ell^2_r} = \sigma_{V_r}.$$

The proof of the next statement is similar to the proof of Proposition 3.5 [4] and therefore it is omitted.

Lemma 6.8.5 ([7])**.** *Assume that J_r is a range function so that $\dim J_r(t) < +\infty$ and the range operator $R_r(t) : J_r(t) \to J_r(t)$ is measurable, for a.e. $t \in \mathbb{T}^n$. Then, $t \mapsto \ker(R_r(t))$, $t \in \mathbb{T}^n$, is a measurable range function.*

Now, by the lemmas 6.8.3–6.8.5 and by Theorem 6.1.1, the next statement follows.

Theorem 6.8.1 ([7])**.** *Let $\alpha \in \ell_r^2$ be a sequence of bounded spectrum. Assume that $V_r \subset H^r$ is a SI space with the range function J_r so that for a.e. $t \in \mathbb{T}^n$, $\dim J_r(t) < +\infty$, and $R_r : J_r \to J_r$ is the corresponding range operator for a shift-preserving operator $L_r : V_r \to V_r$. If $\Lambda_{r,\alpha}$ is an s-eigenvalue for L_r, then for a.e. $t \in \sigma_{V_{r,\alpha}}$ the eigenvulue for $R_r(t)$ is $\lambda_{r,\alpha}(t) = \widehat{\alpha}(t)$. Furthermore,*

$$J_{r,\alpha}(t) = \ker\big(R_r(t) - \lambda_{r,\alpha}(t)1(t)\big) \quad \text{for a.e. } t \in \mathbb{T}^n$$

is a measurable range function of $V_{r,\alpha}$.

The following proposition is stated in the paper [7] as a remark without proof. Therefore, now the proof is performed for the first time in detail.

Proposition 6.8.1 ([7])**.** *Let $\alpha, \beta \in \ell_r^2$, $\alpha \neq \beta$, be sequences of bounded spectrum. Assume that $V_r \subset H^r$ is a SI space, $L_r : V_r \to V_r$ is a shift-preserving operator and $\Lambda_{r,\alpha}$, $\Lambda_{r,\beta}$ are s-eigenvalues for L_r. Then:*

(1) *$V_{r,\alpha}$ is a SI subspace of V_r,*

(2) *$L_r V_{r,\alpha} \subseteq V_{r,\alpha}$,*

(3) *$V_{r,\alpha} \cap V_{r,\beta} = \{\mathbf{0}\}$ if and only if $\widehat{\alpha}(t) \neq \widehat{\beta}(t)$ a.e. in $\sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}}$.*

Proof. The assertions (1) and (2) simply follow from definitions. For (3), if $\sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}} = \emptyset$, then the equivalence holds. Therefore, assume that $\sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}} \neq \emptyset$. Let $V_{r,\alpha} \cap V_{r,\beta} = \{\mathbf{0}\}$. Then, by Proposition 6.1.1 (3), it follows that $J_{V_{r,\alpha}}(t) \cap J_{V_{r,\beta}}(t) = J_{V_{r,\alpha} \cap V_{r,\beta}}(t) = \{\mathbf{0}\}$, i.e.

$$\ker\big(R_r(t) - \lambda_{r,\alpha}(t)1(t)\big) \cap \ker\big(R_r(t) - \lambda_{r,\beta}(t)1(t)\big) = \{\mathbf{0}\} \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Assume that there is a measurable set $A \subseteq \sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}}$ so that $m(A) > 0$ and $\widehat{\alpha}(t) = \widehat{\beta}(t)$ for a.e. $t \in A$. Then, $\ker\big(R_r(t) - \lambda_{r,\alpha}(t)1(t)\big) = \ker\big(R_r(t) - \lambda_{r,\beta}(t)1(t)\big) = \{\mathbf{0}\}$ for a.e. $t \in A$, a contradiction. So, $\widehat{\alpha}(t) \neq \widehat{\beta}(t)$ a.e. in $\sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}}$.

For the opposite implication, let $f \in V_{r,\alpha} \cap V_{r,\beta}$. Then,

$$(\widehat{\alpha} - \widehat{\beta})(t)\mathscr{T}_r f(t) = \{\mathbf{0}\} \quad \text{for a.e. } t \in \mathbb{T}^n,$$

because $L_r f = \Lambda_{r,\alpha} f = \Lambda_{r,\beta} f$. Since $\widehat{\alpha}(t) \neq \widehat{\beta}(t)$ a.e. in $\sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}}$, it implies that $\mathscr{T}_r f(t) = \{\mathbf{0}\}$ for a.e. $t \in \sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}}$. Hence, $f = \mathbf{0}$, since $\sigma_{V_{r,\alpha} \cap V_{r,\beta}} \subseteq \sigma_{V_{r,\alpha}} \cap \sigma_{V_{r,\beta}}$. □

In the rest of this section, a SI space $V_r \subset H^r$ is a FSI space with the range function J_r and $R_r : J_r \to J_r$ is the corresponding range operator for a shift-preserving operator $L_r : V_r \to V_r$.

The definition of s-diagonalization adapted to H^r spaces is given in the next definition.

Definition 6.8.4 ([7])**.** *An operator L_r is said to be s-diagonalizable if there is $c \in \mathbb{N}$ so that Λ_{r,α^k}, $k = 1, \ldots, c$, are s-eigenvalues for L_r and $V_r = V_{r,\alpha^1} \oplus V_{r,\alpha^2} \oplus \cdots \oplus V_{r,\alpha^c}$, where $\oplus$ denotes a direct sum and α^k, $k = 1, \ldots, c$, are sequences of bounded spectrum. $(V_r, L_r, \alpha^1, \ldots, \alpha^c)$ is said to be an s-diagonalization of L_r.*

Theorem 6.8.2 ([7])**.** *If L_r is s-diagonalizable, then $R_r(t)$ is diagonalizable for a.e. $t \in \sigma_{V_r}$.*

Proof. Let $(V_r, L_r, \alpha^1, \dots, \alpha^c)$ be an s-diagonalization for L_r. Using Theorem 6.8.1, it follows that $\lambda_{r,\alpha^k}(t) = \widehat{\alpha^k}(t)$ is an eigenvalue of $R_r(t)$ for a.e. $t \in \sigma_{V_{r,\alpha^k}}$, and $J_{r,\alpha^k}(t) = \ker\big(R_r(t) - \widehat{\alpha^k}(t)1(t)\big)$ is eigenspace for $R_r(t)$ for a.e. $t \in \mathbb{T}^n$, $k = 1, \dots, c$. It is enough to show that $J_r(t) = J_{r,\alpha^1}(t) \oplus \cdots \oplus J_{r,\alpha^c}(t)$ for a.e. $t \in \mathbb{T}^n$.

Obviously, $J_{r,\alpha^1}(t) + \cdots + J_{r,\alpha^c}(t) \subseteq J_r(t)$ for a.e. $t \in \mathbb{T}^n$, since $V_{r,\alpha^k} \subseteq V_r$ implies $J_{r,\alpha^k}(t) \subseteq J_r(t)$ for a.e. $t \in \mathbb{T}^n$, $k = 1, \dots, c$. On the other hand, let $\varphi \in V_r = S_r(\mathscr{A}_{r,m})$. Then, there exist $\varphi_k \in V_{r,\alpha^k}$, $k = 1, \dots, c$, such that $\varphi = \varphi_1 + \cdots + \varphi_c$, and thus $\mathscr{T}_r\varphi(t) = \mathscr{T}_r\varphi_1(t) + \cdots + \mathscr{T}_r\varphi_c(t)$ for a.e. $t \in \mathbb{T}^n$. Therefore, $J_r(t) = \overline{\operatorname{span}}\,\{\mathscr{T}_r f(t) : f \in \mathscr{A}_{r,m}\} \subseteq \overline{J_{r,\alpha^1}(t) + \cdots + J_{r,\alpha^c}(t)}$ for a.e. $t \in \mathbb{T}^n$. Since $\dim_{V_r}(t) < +\infty$ for a.e. $t \in \mathbb{T}^n$, it implies that

$$\overline{J_{r,\alpha^1}(t) + \cdots + J_{r,\alpha^c}(t)} = J_{r,\alpha^1}(t) + \cdots + J_{r,\alpha^c}(t) \quad \text{for a.e. } t \in \mathbb{T}^n.$$

Hence, $J_r(t) = J_{r,\alpha^1}(t) + \cdots + J_{r,\alpha^c}(t)$ for a.e. $t \in \mathbb{T}^n$. By Proposition 6.1.1 (3), $J_r(t) = J_{r,\alpha^1}(t) \oplus \cdots \oplus J_{r,\alpha^c}(t)$ for a.e. $t \in \mathbb{T}^n$, i.e. the sum is direct. □

Theorem 6.8.3 ([7])**.** *Let κ be given by* (6.7.1)*. If the operator $R_r(t)$ is diagonalizable for a.e. $t \in \sigma_{V_r}$, then there are sequences $(\alpha^k)_{k=1}^{\kappa}$ of bounded spectrum so that for a.e. $t \in \mathbb{T}^n$, $J_{r,\alpha^k}(t) = \ker\big(R_r(t) - \widehat{\alpha^k}(t)1(t)\big)$, $k = 1, \dots, \kappa$, are measurable range functions and*

1. *$J_r(t) = J_{r,\alpha^1}(t) \oplus \cdots \oplus J_{r,\alpha^\kappa}(t)$, where $\oplus$ denotes a direct sum;*
2. *the sets $C_k = \big\{t \in \sigma_{V_r} : J_{r,\alpha^k}(t) \neq \{\mathbf{0}\}\big\}$ satisfy $m(C_k) > 0$ and $C_{k+1} \subset C_k$, $k = 1, \dots, \kappa - 1$.*

Proof. (1) Let $\mathrm{D}(V_r) = m$. There are measurable functions $\lambda_{r,1}, \dots, \lambda_{r,\kappa} \in L^\infty(\mathbb{T}^n)$ so that for a.e. $t \in \mathbb{T}^n$,

$$\bigoplus_{k=1}^{\kappa} \ker\big(R_r(t) - \lambda_{r,k}(t)1(t)\big) = J_r(t),$$

by Theorem 6.7.3 and Remark 6.7.1. Indeed, if $t \in A_{m,d}$, then (see the proof of Theorem 6.7.3)

$$\bigoplus_{k=1}^{\kappa} \ker\big(R_r(t) - \lambda_{r,k}(t)1(t)\big) = \bigoplus_{k=1}^{d} \ker\big(R_r(t) - \lambda_{r,m}^{d,k}(t)1(t)\big) \oplus \bigoplus_{k=d+1}^{\kappa} \{\mathbf{0}\} = J_r(t),$$

since for a.e. $t \in \mathbb{T}^n$, $R_r(t)$ is diagonalizable and $\{\lambda_{r,m}^{d,k}(t) : k = 1, \dots, d\}$ is the set of eigenvalues for $R_r(t)$ on $A_{m,d}$. On the other hand, if a.e. $t \notin \sigma_{V_r}$, then $J_r(t) = \{\mathbf{0}\}$ and $\ker\big(R_r(t) - \lambda_{r,k}(t)1(t)\big) = \{\mathbf{0}\}$, $k = 1, \dots, \kappa$.

Finally, since $\lambda_{r,k} \in L^\infty(\mathbb{T}^n)$, it follows that there is a sequence $\alpha^k = (\alpha_q^k)_{q \in \mathbb{Z}^n} \in \ell_r^2$ of bounded spectrum such that $\lambda_{r,k}(t) = \widehat{\alpha^k}(t)$ and $J_{r,\alpha^k}(t) = \ker\big(R_r(t) - \widehat{\alpha^k}(t)1(t)\big)$ is measurable for a.e. $t \in \mathbb{T}^n$, $k = 1, \dots, \kappa$.

(2) By Remark 6.7.1 (2), the statement follows. □

In the following theorem, which represents a generalization of the theorem known as the spectral theorem for shift-preserving operators, the conditions under which the shift-preserving operator is s-diagonalizable are given.

Theorem 6.8.4 ([7]). *If L_r is a normal operator, then L_r is s-diagonalizable and*

$$L_r = \sum_{k=1}^{c} \Lambda_{r,\alpha^k} P_{V_{r,\alpha^k}},$$

where $(V_r, L_r, \alpha^1, \dots, \alpha^c)$ is an s-diagonalization of L_r and $P_{V_{r,\alpha^k}} : V_r \to V_{r,\alpha^k}$, $k = 1, \dots, c$, are the orthogonal projections.

Proof. Since L_r is a normal operator and V_r is a FSI space, using Theorem 6.4.4, it follows that the range operator $R_r(t)$ is normal for a.e. $t \in \mathbb{T}^n$. Therefore, $R_r(t)$ is diagonalizable for a.e. $t \in \mathbb{T}^n$, and thus its eigenspaces are orthogonal.

Let κ be given by (6.7.1). Then, using Theorem 6.8.3, $J_r(t) = J_{r,\alpha^1}(t) \oplus \cdots \oplus J_{r,\alpha^\kappa}(t)$ for a.e. $t \in \mathbb{T}^n$, where J_{r,α^k}, $k = 1, \dots, \kappa$, are measurable range functions. Therefore, by Theorem 6.8.1, $V_{r,\alpha^k} = \{f \in H^r : L_r f = \Lambda_{r,\alpha^k} f\} \neq \{\mathbf{0}\}$, $k = 1, \dots, \kappa$, are SI spaces. Since $J_{r,\alpha^k}(t) \perp J_{r,\alpha^j}(t)$, $k \neq j$, $k, j = 1, \dots, \kappa$, for a.e. $t \in \mathbb{T}^n$, it gives $V_{r,\alpha^k} \perp V_{r,\alpha^j}$, $k \neq j$, $k, j = 1, \dots, \kappa$. Finally, by Proposition 6.1.1 (2), $V_r = V_{r,\alpha^1} \oplus \cdots \oplus V_{r,\alpha^\kappa}$ and thus the operator L_r is s-diagonalizable.

Moreover, if $(V_r, L_r, \alpha^1, \dots, \alpha^c)$ is an s-diagonalization of L_r, then the s-eigenspaces V_{r,α^k}, $k = 1, \dots, c$, are orthogonal and $V_r = V_{r,\alpha^1} \oplus \cdots \oplus V_{r,\alpha^c}$, because the eigenspaces of $R_r(t)$ are orthogonal for a.e. $t \in \mathbb{T}^n$. Hence, the assertion holds. $\square$

Example 6.8.2. *Let $V_r = S_r(\mathscr{A}_{r,m})$ and $E_r(\mathscr{A}_{r,m})$ be a Bessel family for V_r. Then, by Theorem 6.5.2, the associated frame operator for $E_r(\mathscr{A}_{r,m})$ is shift-preserving. However, since it is self-adjoint, by Theorem 6.8.4, it follows that it is s-diagonalizable.*

Theorem 6.8.5 ([7]). *Let $L_r : V_r \to V_r$ be a normal operator. Then,*

(1) *the operators L_r and (its adjoint) L_r^* are s-diagonalizable;*

(2) *if $\Lambda_{r,\alpha}$ is an s-eigenvalue for L_r, then (its adjoint) $\Lambda_{r,\alpha}^*$ is an s-eigenvalue for L_r^* and $V_{r,\alpha} = \{f \in H^r : L_r^* f = \Lambda_{r,\alpha}^* f\}$. Furthermore, $\Lambda_{r,\alpha}^* = \Lambda_{r,\widetilde{\alpha}}$, where $\widetilde{\alpha} = (\widetilde{\alpha_q})_{q \in \mathbb{Z}^n} \in \ell_r^2$ and $\widetilde{\alpha_q} = \overline{\alpha_{-q}}$, $q \in \mathbb{Z}^n$;*

(3) *if $(V_r, L_r, \alpha^1, \dots, \alpha^c)$ is an s-diagonalization for L_r, then $(V_r, L_r^*, \widetilde{\alpha^1}, \dots, \widetilde{\alpha^c})$ is an s-diagonalization for L_r^*.*

Proof. (1) Since L_r is a normal operator, its adjoint operator L^* is also normal. Therefore, by Theorem 6.8.4, the operators L_r and L_r^* are s-diagonalizable.

(2) Since

$$\overline{\widehat{\alpha}(t)} = \sum_{q \in \mathbb{Z}^n} \overline{\alpha_q}\, e^{2\pi i \langle t,q \rangle} = \sum_{q \in \mathbb{Z}^n} \overline{\alpha_{-q}}\, e^{-2\pi i \langle t,q \rangle},$$

set $\widetilde{\alpha_q} = \overline{\alpha_{-q}}$, $q \in \mathbb{Z}^n$. Obviously, $\widetilde{\alpha} = (\widetilde{\alpha_q})_{q \in \mathbb{Z}^n} \in \ell_r^2$. Let $f_1, f_2 \in V_r$. Then,

$$\langle \Lambda_{r,\alpha} f_1, f_2 \rangle_{H^r} = \langle f_1, \Lambda_{r,\widetilde{\alpha}} f_2 \rangle_{H^r},$$

i.e. $\Lambda_{r,\alpha}^* = \Lambda_{r,\widetilde{\alpha}}$. Furthermore,

$$\ker(L_r^* - \Lambda_{r,\widetilde{\alpha}}) = \ker((L_r - \Lambda_{r,\alpha})^*) = \ker(L_r - \Lambda_{r,\alpha}) = V_{r,\alpha} \neq \{\mathbf{0}\},$$

because $L_r - \Lambda_{r,\alpha}$ is a normal operator. Hence, the assertion holds.

(3) Assume that $(V_r, L_r, \alpha^1, \dots, \alpha^c)$ is an s-diagonalization for L_r. Then, by (2),

$$\ker(L_r^* - \Lambda_{r,\widetilde{\alpha^k}}) = V_{r,\alpha^k}, \quad k = 1, \dots, c.$$

Thus, $V_r = V_{r,\alpha^1} \oplus \cdots \oplus V_{r,\alpha^c}$ is the decomposition on s-eigenspaces of L_r^*. □

6.9 Dynamical sampling

In this section, assume that $V_r = S_r(f_1, \dots, f_m) \subset H^r$ is a FSI space with the range function J_r and $R_r : J_r \to J_r$ is the corresponding range operator for a shift-preserving operator $L_r : V_r \to V_r$, R_r^* and L_r^* are corresponding adjoint operators, respectively, $E = \{1, 2, \dots, m-1\}$, $J = \{0, 1, \dots, s\}$, and $V_{r,\beta} = \{f \in H^r : L_r^* f = \Lambda_{r,\beta} f\}$, i.e. $V_{r,\beta} = \ker(L_r^* - \Lambda_{r,\beta})$, where $\beta \in \ell_r^2$ is a sequence of bounded spectrum so that $\Lambda_{r,\beta}$ is an s-eigenvalue for L_r^*.

The next theorem is Theorem 3.2 from [5] adapted to observed spaces.

Theorem 6.9.1 ([7]). *Assume that $\widehat{\beta}(t)$ is an eigenvalue for $R_r^*(t)$ for a.e. $t \in \sigma_{V_{r,\beta}}$, and let $J_{r,\beta}(t) = \ker\big(R_r^*(t) - \widehat{\beta}(t) 1(t)\big)$. If*

$$\{(R_r(t))^k (\mathscr{T}_r \varphi_j(t)) : \varphi_j \in V_r,\ j \in J,\ k \in E\}$$

is a frame for $J_r(t)$ with frame bounds $A, B > 0$ for a.e. $t \in \sigma_{V_{r,\beta}}$, then

$$\{P_{J_{r,\beta}(t)}(\mathscr{T}_r \varphi_j(t)) : \varphi_j \in V_r,\ j \in J\}$$

is a frame for $J_{r,\beta}(t)$ with frame bounds $\frac{A}{C(t)}$, $\frac{B}{C(t)}$, where $C(t) = \sum_{k \in E} |\widehat{\beta}(t)|^{2k}$, for a.e. $t \in \sigma_{V_{r,\beta}}$.

Proof. For every $\varphi \in V_{r,\beta}$,

$$\begin{aligned}
\sum_{k\in E}\sum_{j\in J} \big|\big\langle \mathscr{T}_r\varphi(t), (R_r(t))^k(\mathscr{T}_r\varphi_j(t))\big\rangle_{\ell_r^2}\big|^2 &= \sum_{k\in E}\sum_{j\in J} \big|\big\langle (R_r^*(t))^k(\mathscr{T}_r\varphi(t)), \mathscr{T}_r\varphi_j(t)\big\rangle_{\ell_r^2}\big|^2 \\
&= \sum_{k\in E}\sum_{j\in J} \big|\big\langle (\widehat{\beta}(t))^k(\mathscr{T}_r\varphi(t)), \mathscr{T}_r\varphi_j(t)\big\rangle_{\ell_r^2}\big|^2 \\
&= \sum_{k\in E} |\widehat{\beta}(t)|^{2k} \sum_{j\in J} \big|\big\langle \mathscr{T}_r\varphi(t), P_{J_{r,\beta}(t)}(\mathscr{T}_r\varphi_j(t))\big\rangle_{\ell_r^2}\big|^2.
\end{aligned}$$

Therefore, the statement follows. □

Theorem 6.9.2 ([7]). *If the set $\{L_r^k \varphi_j : \varphi_j \in V_r,\ j \in J,\ k \in E\}$ is a frame generator for V_r with frame bounds $A, B > 0$, then the set*

$$\{P_{V_{r,\beta}} \varphi_j : \varphi_j \in V_r,\ j \in J\}$$

is a frame generator for $V_{r,\beta}$ with frame bounds $A / \sum_{k=0}^{m-1} \|L_r\|^{2k}$ and B.

Proof. Let the set $\{L_r^k \varphi_j : \varphi_j \in V_r,\ j \in J,\ k \in E\}$ be a frame generator for V_r with frame bounds $A, B > 0$. Then, for a.e. $t \in \mathbb{T}^n$,

$$\{\mathscr{T}_r\big(L_r^k \varphi_j\big)(t) : \varphi_j \in V_r,\ j \in J,\ k \in E\} = \{(R_r(t))^k\big(\mathscr{T}_r\varphi_j(t)\big) : \varphi_j \in V_r,\ j \in J,\ k \in E\}$$

is a frame for $J_r(t)$ with the frame bounds $A, B > 0$, by Theorem 6.2.1 (1) and the equality (6.4.6). Further, let $\Lambda_{r,\beta}$ be an s-eigenvalue for L_r^*. Then, by the theorems 6.4.4 and 6.8.1, $\widehat{\beta}(t)$ is an eigenvalue for $R_r^*(t)$ for a.e. $t \in \sigma_{V_{r,\beta}}$. Now, by Theorem 6.9.1,

$$\big\{P_{J_{r,\beta}(t)}\big(\mathscr{T}_r\varphi_j(t)\big) : \varphi_j \in V_r,\ j \in J\big\}$$

is a frame of $J_{r,\beta}(t)$ with frame bounds $A/\sum_{k=0}^{m-1}|\widehat{\beta}(t)|^{2k}$ and $B/\sum_{k=0}^{m-1}|\widehat{\beta}(t)|^{2k}$ for a.e. $t \in \sigma_{V_{r,\beta}}$. Finally, since

$$1 \leqslant C(t) = \sum_{k=0}^{m-1}|\widehat{\beta}(t)|^{2k} \leqslant \sum_{k=0}^{m-1}\|R_r(t)\|^{2k} \leqslant \sum_{k=0}^{m-1}\|L_r\|^{2k},$$

the assertion follows, by Lemma 6.1.4 (see Remark 6.1.1) and Theorem 6.2.1 (1). □

The following statement is Theorem 3.5 from [5] adapted to observed spaces and operators.

Theorem 6.9.3 ([5])**.** *Let $\alpha^1, \ldots, \alpha^c$ be sequences of bounded spectrum and R_r be a normal range operator such that*

$$R_r(t) = \sum_{l=1}^{c}\widehat{\alpha^l}(t)P_{J_{r,\alpha^l}(t)} \quad \textit{for a.e. } t \in \mathbb{T}^n.$$

If for every $l = 1, \ldots, c$ and for a.e. $t \in \mathbb{T}^n$,

$$\{P_{J_{r,\alpha^l}(t)}(\mathscr{T}_r f_j(t)) : f_j \in V_r,\ j \in J\}$$

is a frame for $J_{r,\alpha^l}(t)$ with frame bounds $A_k, B_k > 0$, then for a.e. $t \in \sigma_{V_r}$,

$$\{(R_r(t))^k(\mathscr{T}_r f_j(t)) : f_j \in V_r,\ j \in J,\ k \in E\}$$

is a frame for $J_r(t)$ with frame bounds

$$A\bigg(\frac{c}{\gamma(t)}\sum_{l=0}^{c-1}\binom{c-1}{l}^2\|R_r(t)\|^{2l}\bigg)^{-1} \quad \textit{and} \quad B\bigg(c\sum_{k=0}^{m-1}\|R_r(t)\|^{2k}\bigg),$$

where $\gamma(t) = \min_{1\leqslant l\leqslant c}\prod_{p=1,p\neq l}^{c}|\widehat{a^l}(t) - \widehat{a^p}(t)| > 0$ and $A = \min_{1\leqslant l\leqslant c} A_l$, $B = \min_{1\leqslant l\leqslant c} B_l$.

Under additional assumptions the equivalence in Theorem 6.9.2 holds.

Definition 6.9.1 ([5], [7])**.** *A shift-preserving operator L_r is said to have the spectral property if for a.e. $t \in \sigma_{V_r}$ there is $C > 0$ so that $|\lambda_r' - \lambda_r| \geqslant C$ for all $\lambda_r' \neq \lambda_r$, where $\lambda_r, \lambda_r' \in \mathfrak{R}_{\lambda_r}(t)$.*

Finally, the problem of dynamical sampling for shift-preserving operators L_r on $V_r = S_r(f_1, \ldots, f_m) \subset H^r$ is solved by the following theorem.

Theorem 6.9.4 ([7])**.** *Let a shift-preserving operator L_r be a normal operator which has the spectral property. Then, the set*

$$\{L_r^k\varphi_j : \varphi_j \in V_r,\ j \in J,\ k \in E\}$$

is a frame generator of V_r if and only if

$$\{P_{V_{r,\beta}}\varphi_j : \varphi_j \in V_r,\ j \in J\}$$

is a frame generator of $V_{r,\beta}$ with the same frame bounds for every s-eigenvalue $\Lambda_{r,\beta}$ of L_r^*.

Proof. The necessary condition is proved by Theorem 6.9.2. Therefore, only the implication in the other direction should be shown.

Let L_r be a normal operator which has the spectral property. Then, by Theorem 6.8.5, L^* is an s-diagonalizable operator. Therefore, one can construct an s-diagonalization $(V_r, L_r^*, \beta^1, \ldots, \beta^\kappa)$ of L_r^* so that $\sigma_{V_{r,\beta^{l+1}}} \subset \sigma_{V_{r,\beta^l}}$ for $l = 1, \ldots, \kappa - 1$, and for a.e. $t \in \mathbb{T}^n$,

$$|\widehat{\beta^l}(t) - \widehat{\beta^p}(t)| \geqslant C > 0, \quad l \neq p, \quad l, p = 1, \ldots, \kappa, \tag{6.9.1}$$

since L_r has the spectral property (see Theorem 6.8.3 and Proposition 6.8.1 (3)), where κ is given by (6.7.1).

Assume that $\{P_{V_{r,\beta^l}}\varphi_j : \varphi_j \in V_r,\ j \in J\}$ is a frame generator of V_{r,β^l}, $l = 1, \ldots, \kappa$, with frame bounds $A, B > 0$. For a.e. $t \in \mathbb{T}^n$, by Theorem 6.2.1,

$$\{\mathscr{T}_r(P_{V_{r,\beta^l}}\varphi_j)(t) : \varphi_j \in V_r,\ j \in J\}$$

is a frame of $J_{r,\beta^l}(t)$, $l = 1, \ldots, \kappa$, with the same frame bounds. By Lemma 6.1.4 (see Remark 6.1.1), for a.e. $t \in \mathbb{T}^n$,

$$\{P_{J_{r,\beta^l}(t)}(\mathscr{T}_r\varphi_j(t)) : \varphi_j \in V_r,\ j \in J\}$$

is a frame of $J_{r,\beta^l}(t)$, $l = 1, \ldots, \kappa$, with the same frame bounds. Set $A_l = \sigma_{V_{r,\beta^l}} \backslash \sigma_{V_{r,\beta^{l+1}}}$, $l = 1, \ldots, \kappa - 1$, and $A_\kappa = \sigma_{V_{r,\beta^\kappa}}$. Then, $\big(\mathbb{T}^n \backslash \sigma_{V_r}\big) \cup \bigcup_{l=1}^{\kappa} A_l = \mathbb{T}^n$.

Fix $\widetilde{l} \in \{1, \ldots, \kappa\}$. Then, $\{P_{J_{r,\beta^l}(t)}(\mathscr{T}_r\varphi_j(t)) : \varphi_j \in V_r,\ j \in J\}$ is a frame of J_{r,β^l} for a.e. $t \in A_l$, $l = 1, \ldots, \widetilde{l}$. By Theorem 6.9.3,

$$\{(R_r(t))^k(\mathscr{T}_r\varphi_j(t)) : \varphi_j \in V_r,\ j \in J,\ k \in E\}$$

is a frame of $J_r(t)$ with frame bounds

$$A\bigg(\frac{\kappa}{\gamma(t)}\sum_{l=0}^{\widetilde{l}-1}\binom{\widetilde{l}-1}{l}^2\|R_r(t)\|^{2l}\bigg)^{-1} \quad \text{and} \quad B\bigg(\widetilde{l}\sum_{k=0}^{m-1}\|R_r(t)\|^{2k}\bigg),$$

where $\gamma(t) = \min\limits_{1\leqslant l\leqslant \widetilde{l}} \prod\limits_{l=1, l\neq p}^{\widetilde{l}} |\widehat{\beta^l}(t) - \widehat{\beta^p}(t)|^2$, for a.e. $t \in A_{\widetilde{l}}$.

Without loss of generality, one can assume that $C < 1$ in (6.9.1). Then, for a.e. $t \in A_{\widetilde{l}}$, $C^{2\kappa} \leqslant C^{2\widetilde{l}} \leqslant \gamma(t)$ and

$$A\bigg(\frac{\kappa}{C^{2\kappa}}\sum_{l=0}^{\kappa-1}\binom{\kappa-1}{l}^2\|L_r\|^{2l}\bigg)^{-1} \leqslant A\bigg(\frac{\kappa}{\gamma(t)}\sum_{l=0}^{\widetilde{l}-1}\binom{\widetilde{l}-1}{l}^2\|R_r(t)\|^{2l}\bigg)^{-1},$$

$$B\bigg(\widetilde{l}\sum_{k=0}^{m-1}\|R_r(t)\|^{2k}\bigg) \leqslant B\bigg(\kappa\sum_{k=0}^{m-1}\|L_r\|^{2k}\bigg).$$

Note that, these frame bounds are the same for all sets $A_{\widetilde{l}}$, $\widetilde{l} = 1, \ldots, \kappa$. Thus,

$$\{(R_r(t))^k(\mathscr{T}_r\varphi_j(t)) : \varphi_j \in V_r,\ j \in J,\ k \in E\}$$

is a frame of $J_r(t)$ for a.e. $t \in \mathbb{T}^n$. Therefore, $\{L_r^k \varphi_j : \varphi_j \in V_r,\ j \in J,\ k \in E\}$ is a frame generator of V_r with the frame bounds

$$A\biggl(\frac{\kappa}{C^{2\kappa}} \sum_{l=0}^{\kappa-1} \binom{\kappa-1}{l}^2 \|L_r\|^{2l}\biggr)^{-1}, \qquad B\biggl(\kappa \sum_{k=0}^{m-1} \|L_r\|^{2k}\biggr),$$

by (6.4.6) and Theorem 6.2.1. Hence, the statement holds. □

6.10 Products in shift-invariant spaces

In this and the last section, results of the paper [8] will be presented. The first result is a consequence of Theorem 4.6.1.

Proposition 6.10.1 ([8]). *Let*

$$\varphi_1 = \sum_{q\in\mathbb{Z}^n} \alpha_{q,1} T_q f_1 \quad \text{and} \quad \varphi_2 = \sum_{q\in\mathbb{Z}^n} \alpha_{q,2} T_q f_2,$$

with $(\alpha_{q,1})_{q\in\mathbb{Z}^n} \in \ell^1_{r_1}$, $(\alpha_{q,2})_{q\in\mathbb{Z}^n} \in \ell^2_{r_2}$, $f_1 \in H^{r_1}$, $f_2 \in H^{r_2} \cap \mathscr{F}^{-1}(L^\infty)$ *and* $r_1 + r_2 \geqslant 0$. *Then,*

$$\varphi = \varphi_1 * \varphi_2 \in V_r(f_1 * f_2),$$

where $r \leqslant \min\{r_1, r_2\}$, *i.e.*

$$\varphi = \sum_{q\in\mathbb{Z}^n} \alpha_q T_q(f_1 * f_2), \quad (\alpha_q)_{q\in\mathbb{Z}^n} \in \ell^2_r, \quad \alpha_q = \sum_{p\in\mathbb{Z}^n} \alpha_{q-p,1}\alpha_{p,2},\ q \in \mathbb{Z}^n.$$

Proof. First, using the theorems 3.2.3 (2), 3.4.3 (3) and 4.6.1, it follows that

$$\begin{aligned}\widehat{\varphi} = \widehat{\varphi_1 * \varphi_2} = \widehat{\varphi_1}\widehat{\varphi_2} &= \sum_{q\in\mathbb{Z}^n} \alpha_{q,1} M_{-q}\widehat{f_1} \sum_{q\in\mathbb{Z}^n} \alpha_{q,2} M_{-q}\widehat{f_2} \\ &= \widehat{f_1}\widehat{f_2} \sum_{q\in\mathbb{Z}^n} \alpha_{q,1}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \sum_{q\in\mathbb{Z}^n} \alpha_{q,2}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} = \widehat{f_1}\widehat{f_2} \sum_{q\in\mathbb{Z}^n} \alpha_q\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle},\end{aligned} \tag{6.10.1}$$

where $(\alpha_q)_{q\in\mathbb{Z}^n} \in \ell^2_r$, $\alpha_q = \sum_{p\in\mathbb{Z}^n} \alpha_{q-p,1}\alpha_{p,2}$, $q \in \mathbb{Z}^n$. Further,

$$\begin{aligned}\int_{\mathbb{R}^n} |\widehat{f_1}(t)\widehat{f_2}(t)|^2 \mu_r^2(t)\,\mathrm{d}t &\leqslant \|\widehat{f_2}\|^2_{L^\infty} \int_{\mathbb{R}^n} |\widehat{f_1}(t)|^2 \mu_r^2(t)\,\mathrm{d}t = \|\widehat{f_2}\|^2_{L^\infty} \|f_1\|^2_{H^r} \\ &\leqslant \|\widehat{f_2}\|^2_{L^\infty} \|f_1\|^2_{H^{r_1}} < +\infty,\end{aligned}$$

since $r_1 \geqslant r$. Thus, $\widehat{f_1}\widehat{f_2} \in L^2_r$, i.e. $f_1 * f_2 \in H^r$, by Lemma 3.5.2 and Theorem 3.2.3 (2). Applying the inverse Fourier transform to (6.10.1), it implies that

$$\varphi(x) = (\varphi_1 * \varphi_2)(x) = (f_1 * f_2)(x) * \sum_{q\in\mathbb{Z}^n} \alpha_q \delta_q(x) = \sum_{q\in\mathbb{Z}^n} \alpha_q T_q(f_1 * f_2)(x), \quad x \in \mathbb{R}^n,$$

by the theorems 3.2.3 (2), 3.4.3 (3) and by Example 3.4.2. Hence, the statement holds. □

In order to introduce the property of compatible coefficient estimates, the following conditions are required. Let $(\alpha_{q,1}^k)_{q\in\mathbb{Z}^n}$, $(\alpha_{q,2}^l)_{q\in\mathbb{Z}^n}$, $k=1,\dots,s_1$, $l=1,\dots,s_2$, and $\Lambda_1^k,\Lambda_2^l\subset\mathbb{Z}^n$, $\Lambda_1^k\cap(-\Lambda_2^l)=\emptyset$, $k=1,\dots,s_1$, $l=1,\dots,s_2$, be such that for every $k=1,\dots,s_1$ and every $l=1,\dots,s_2$,

$$\sum_{q\in\Lambda_1^k}|\alpha_{q,1}^k|^2\mu_{-2a_1}(q)<+\infty,\qquad \sum_{q\in\mathbb{Z}^n\setminus\Lambda_1^k}|\alpha_{q,1}^k|^2\mu_{2b_1}(q)<+\infty, \tag{6.10.2}$$

$$\sum_{q\in\Lambda_2^l}|\alpha_{q,2}^l|^2\mu_{-2a_2}(q)<+\infty,\qquad \sum_{q\in\mathbb{Z}^n\setminus\Lambda_2^l}|\alpha_{q,2}^l|^2\mu_{2b_2}(q)<+\infty, \tag{6.10.3}$$

for some $b_1\geqslant a_2\geqslant 0$, $b_2\geqslant a_1\geqslant 0$. Moreover, let for all $k=1,\dots,s_1$, $l=1,\dots,s_2$ and every $q\in\mathbb{Z}^n$ there are $C>0$ and $a\geqslant 1$ so that

$$\gamma_1^{k,l}(q)=\operatorname{card}\{p\in\mathbb{Z}^n\ :\ q-p\in\Lambda_2^l\ \wedge\ p\in\Lambda_1^k\}\leqslant C|q|^a. \tag{6.10.4}$$

New terminology, such as compatible sequences and compatible coefficient estimates are given in the following definition.

Definition 6.10.1 ([8]). (1) *Functions $v_1,v_2\in\mathscr{P}'$ are said to have compatible coefficient estimates if for their sequences of coefficients* (6.10.2)–(6.10.4) *hold.*

(2) *Functions $f_1,f_2\in\mathscr{D}'$ in a neighborhood of $x_0\in\mathbb{R}^n$ have compatible coefficient estimates if for some $\phi\in\mathscr{D}(\mathbb{T}^n_{x_0,\theta})$, $(\phi f_1)_{pe}$ and $(\phi f_2)_{pe}$ have Fourier expansions such that* (6.10.2)–(6.10.4) *hold.*

(3) *Sequences $(\alpha_{q,1}^k)_{q\in\mathbb{Z}^n}$ and $(\alpha_{q,2}^l)_{q\in\mathbb{Z}^n}$, $k=1,\dots,s_1$, $l=1,\dots,s_2$, are said to be compatible sequences if* (6.10.2)–(6.10.4) *hold.*

A new result for the product of periodic distributions (i.e. elements of the space $\mathscr{P}'$) is given in the following assertion, which is very significant for the following results.

Theorem 6.10.1 ([8]). *Let $v_1,v_2\in\mathscr{P}'$, i.e.*

$$v_1=\sum_{k=1}^{s_1}\sum_{q\in\mathbb{Z}^n}\alpha_{q,1}^k\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle},\quad v_2=\sum_{l=1}^{s_2}\sum_{q\in\mathbb{Z}^n}\alpha_{q,2}^l\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle},$$

where $\sum_{q\in\mathbb{Z}^n}|\alpha_{q,1}^k|^2\mu_{-2\tau_1}(q)<+\infty$ and $\sum_{q\in\mathbb{Z}^n}|\alpha_{q,2}^l|^2\mu_{-2\tau_2}(q)<+\infty$, for some $\tau_1,\tau_2>0$ and all $k=1,\dots,s_2$, $l=1,\dots,s_2$. If v_1 and v_2 have compatible coefficient estimates, then there is a $\tau\in\mathbb{R}$ so that $v_1v_2\in\mathscr{P}'^\tau$.

Proof. First, it is not difficult to see that if

$$\gamma_2^{l,k}(q)=\operatorname{card}\{p\in\mathbb{Z}^n\ :\ q-p\in\Lambda_1^k\ \wedge\ p\in\Lambda_2^l\},$$

then $\gamma_1^{k,l}(q)=\gamma_2^{l,k}(q)$.

Let $v_1,v_2\in\mathscr{P}'$ have compatible coefficient estimates. Since the general case is just a repetition of the following procedure, the indices k and l can be omitted (i.e. let $s_1=1$ and $s_2=1$). Therefore,

$$\begin{aligned} v_1v_2&=\Bigg(\sum_{q\in\Lambda_1}+\sum_{q\in\mathbb{Z}^n\setminus\Lambda_1}\Bigg)\alpha_{q,1}\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle}\cdot\Bigg(\sum_{q\in\Lambda_2}+\sum_{q\in\mathbb{Z}^n\setminus\Lambda_2}\Bigg)\alpha_{q,2}\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle}\\ &=v_1^1v_2^1+v_1^1v_2^2+v_1^2v_2^1+v_1^2v_2^2. \end{aligned} \tag{6.10.5}$$

Suppose that $2\tau \geqslant \max\big\{4a(a_1+a_2)+2a+n+1, 2a_1+n+1, 2a_2+n+1\big\}$. It should be shown that every term of the sum (6.10.5) is finite.

For the first term of the sum (6.10.5),

$$v_1^1 v_2^1 = \sum_{q\in\mathbb{Z}^n} \alpha_q^{11}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}, \quad \text{where} \quad \alpha_q^{11} = \sum_{\substack{q-p\in\Lambda_1\\ p\in\Lambda_2}} \alpha_{q-p,1}\alpha_{p,2}, \quad q\in\mathbb{Z}^n,$$

using (6.10.4) and knowing that for $a \geqslant 1$, $|q|^a \leqslant \mu_a(q)$ holds, it follows that

$$\begin{aligned}
\sum_{q\in\mathbb{Z}^n} |\alpha_q^{11}|^2 \mu_\tau^{-2}(q) &\leqslant \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\Lambda_2}} |\alpha_{q-p,1}||\alpha_{p,2}|\Bigg)^2 \mu_\tau^{-2}(q)\\
&= \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\Lambda_2}} |\alpha_{q-p,1}|\mu_{a_1}^{-1}(q-p)|\alpha_{p,2}|\mu_{a_2}^{-1}(p)\mu_{a_1}(q-p)\mu_{a_2}(p)\Bigg)^2 \mu_\tau^{-2}(q)\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\Lambda_2}} |\alpha_{q-p,1}|\mu_{-a_1}(q-p)|\alpha_{p,2}|\mu_{-a_2}(p)\Bigg)^2 \mu_{4a(a_1+a_2)+2a-2\tau}(q)\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\Lambda_2}} |\alpha_{q-p,1}|^2\mu_{-2a_1}(q-p)\Bigg)\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\Lambda_2}} |\alpha_{p,2}|^2\mu_{-2a_2}(p)\Bigg)\mu_{n+1}^{-1}(q)\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\mu_{n+1}^{-1}(q) < +\infty,
\end{aligned}$$

since for $p\in\Lambda_2$ and $(q-p)\in\Lambda_1$,

$$\mu_{a_1}(q-p) \leqslant \mu_{a_1}\big((q_1+|q|^a,\dots,q_n+|q|^a)\big) \leqslant C\mu_{2aa_1}(q), \quad \mu_{a_2}(p) \leqslant C\mu_{2aa_2}(q),$$

again using (6.10.4).

For the second term of the sum (6.10.5),

$$v_1^1 v_2^2 = \sum_{q\in\mathbb{Z}^n} \alpha_q^{12}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}, \quad \text{where} \quad \alpha_q^{12} = \sum_{\substack{q-p\in\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} \alpha_{q-p,1}\alpha_{p,2}, \quad q\in\mathbb{Z}^n,$$

it follows that

$$\begin{aligned}
\sum_{q\in\mathbb{Z}^n} |\alpha_q^{12}|^2 \mu_\tau^{-2}(q) &\leqslant \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{q-p,1}|\mu_{-a_1}(q-p)|\alpha_{p,2}|\mu_{b_2}(p)\frac{\mu_{a_1}(q-p)}{\mu_{b_2}(p)}\Bigg)^2 \mu_\tau^{-2}(q)\\
&\leqslant \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{q-p,1}|\mu_{-a_1}(q-p)|\alpha_{p,2}|\mu_{b_2}(p)\Bigg)^2 \mu_{2a_1-2\tau}(q)\\
&\leqslant \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{q-p,1}|^2\mu_{-2a_1}(q-p)\Bigg)\Bigg(\sum_{\substack{q-p\in\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{p,2}|^2\mu_{2b_2}(p)\Bigg)\mu_{2a_1-2\tau}(q)\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\mu_{2a_1-2\tau}(q) < +\infty,
\end{aligned}$$

because $\mu_{a_1}(q-p) \leqslant C\mu_{a_1}(q)\mu_{a_1}(p)$ and $b_2 \geqslant a_1 \geqslant 0$.

Next, Peetre's inequality (3.5.5) for $t = p$, $x = q$ and $r = a_2 \geqslant 0$ gives

$$\mu_{a_2}(p) \leqslant C\mu_{a_2}(q-p)\mu_{a_2}(q).$$

Thus, the estimate for

$$v_1^2 v_2^1 = \sum_{q\in\mathbb{Z}^n} \alpha_q^{21}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}, \quad \text{where} \quad \alpha_q^{21} = \sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\Lambda_2}} \alpha_{q-p,1}\alpha_{p,2}, \quad q\in\mathbb{Z}^n,$$

simply follows:

$$\begin{aligned}
\sum_{q\in\mathbb{Z}^n} |\alpha_q^{21}|^2\mu_\tau^{-2}(q) &\leqslant \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\Lambda_2}} |\alpha_{q-p,1}|\mu_{b_1}(q-p)|\alpha_{p,2}|\mu_{a_2}^{-1}(p)\frac{\mu_{a_2}(p)}{\mu_{b_1}(q-p)}\Bigg)^2\mu_\tau^{-2}(q)\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\Lambda_2}} |\alpha_{q-p,1}|^2\mu_{b_1}^2(q-p)\Bigg)\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\Lambda_2}} |\alpha_{p,2}|^2\mu_{a_2}^{-2}(p)\Bigg)\frac{\mu_{a_2}^2(q)}{\mu_\tau^2(q)}\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\mu_{n+1}^{-1}(q) < +\infty,
\end{aligned}$$

since $b_1 \geqslant a_2 \geqslant 0$.

Finally, for

$$v_1^2 v_2^2 = \sum_{q\in\mathbb{Z}^n} \alpha_q^{22}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}, \quad \text{where} \quad \alpha_q^{22} = \sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} \alpha_{q-p,1}\alpha_{p,2}, \quad q\in\mathbb{Z}^n,$$

hold

$$\begin{aligned}
\sum_{q\in\mathbb{Z}^n} |\alpha_q^{22}|^2\mu_\tau^{-2}(q) &\leqslant \sum_{q\in\mathbb{Z}^d}\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{q-p,1}|\mu_{b_1}(q-p)|\alpha_{p,2}|\mu_{b_2}(p)\mu_{b_1}^{-1}(q-p)\mu_{b_2}^{-1}(p)\Bigg)^2\mu_\tau^{-2}(q)\\
&\leqslant \sum_{q\in\mathbb{Z}^d}\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{q-p,1}|\mu_{b_1}(q-p)|\alpha_{p,2}|\mu_{b_2}(p)\Bigg)^2\mu_\tau^{-2}(q)\\
&\leqslant \sum_{q\in\mathbb{Z}^n}\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{q-p,1}|^2\mu_{2b_1}(q-p)\Bigg)\Bigg(\sum_{\substack{q-p\in\mathbb{Z}^n\setminus\Lambda_1\\ p\in\mathbb{Z}^n\setminus\Lambda_2}} |\alpha_{p,2}|^2\mu_{2b_2}(p)\Bigg)\mu_\tau^{-2}(q)\\
&\leqslant C\sum_{q\in\mathbb{Z}^n}\mu_\tau^{-2}(q) < +\infty,
\end{aligned}$$

since $b_1, b_2 \geqslant 0$.

Hence, there is a $\tau\in\mathbb{R}$ so that $v_1v_2 \in \mathscr{P}'^\tau$. □

The result of Theorem 6.10.1 can be applied to the product (convolution) of elements of SI spaces.

Theorem 6.10.2 ([8])**.** *Let* $\varphi_1 \in V_{r_1}(f_1^1,\dots,f_1^{s_1})$ *and* $\varphi_2 \in V_{r_2}(f_2^1,\dots,f_2^{s_2})$ *such that*

$$\varphi_1 = \sum_{k=1}^{s_1}\sum_{q\in\mathbb{Z}^n}\alpha_{q,1}^k T_q f_1^k, \quad \varphi_2 = \sum_{l=1}^{s_2}\sum_{q\in\mathbb{Z}^n}\alpha_{q,2}^l T_q f_2^l,$$

where $r_1, r_2 \geqslant 0$. *If* $(\alpha^k_{q,1})_{q\in\mathbb{Z}^n}$ *and* $(\alpha^l_{q,2})_{q\in\mathbb{Z}^n}$, $k = 1,\ldots,s_1$, $l = 1,\ldots,s_2$, *are compatible sequences, then there is an* $r \in \mathbb{R}$ *so that for* $f^k_1, f^l_2 \in V_r \cap \mathscr{V}_r$, $k = 1,\ldots,s_1$, $l = 1,\ldots,s_2$,

$$\varphi_1 * \varphi_2 \in V_r\big(f^k_1 * f^l_2,\ k = 1,\ldots,s_1,\ l = 1,\ldots,s_2\big).$$

More precisely,

$$\varphi_1 * \varphi_2 = \sum_{k=1}^{s_1}\sum_{l=1}^{s_2}\sum_{\substack{q\in\mathbb{Z}^n\\ q-p\in\mathbb{Z}^n}} \alpha^k_{q-p,1}\alpha^l_{p,2} T_q(f^k_1 * f^l_2).$$

Proof. The general case is just a repetition of the following procedure so the indices k and l can be omitted (i.e. let $s_1 = 1$ and $s_2 = 1$). Therefore, applying the Fourier transform to φ_1 and φ_2 gives

$$\widehat{\varphi_1}(t) = \widehat{f_1}(t)v_1(t), \quad \widehat{\varphi_2}(t) = \widehat{f_2}(t)v_2(t), \quad t \in \mathbb{R}^n,$$

where

$$v_1(t) = \sum_{q\in\mathbb{Z}^n} \alpha_{q,1}\,\mathrm{e}^{-2\pi i\langle t,q\rangle}, \quad v_2(t) = \sum_{q\in\mathbb{Z}^n} \alpha_{q,2}\,\mathrm{e}^{-2\pi i\langle t,q\rangle},$$

by Theorem 3.4.3 (3). Now, since v_1 and v_2 have compatible coefficient estimates, Theorem 6.10.1 gives a $\tau \in \mathbb{R}$ so that coefficients $\alpha_q = \sum_{p\in\mathbb{Z}^n} \alpha_{q-p,1}\alpha_{p,2}$, $q \in \mathbb{Z}^n$, satisfy

$$\sum_{q\in\mathbb{Z}^n} |\alpha_q|^2 \mu_{-2\tau}(q) < +\infty.$$

This implies

$$\widehat{\varphi_1}(t)\widehat{\varphi_2}(t) = \widehat{f_1}(t)\widehat{f_2}(t)\sum_{q\in\mathbb{Z}^n} \alpha_q\,\mathrm{e}^{-2\pi i\langle t,q\rangle}, \quad t \in \mathbb{R}^n,$$

and thus

$$(\varphi_1 * \varphi_2)(x) = (f_1 * f_2)(x) * \sum_{q\in\mathbb{Z}^n} \alpha_q\delta_q(x) = \sum_{q\in\mathbb{Z}^n} \alpha_q T_q(f_1 * f_2)(x), \quad x \in \mathbb{R}^n,$$

by the theorems 3.2.3 (2), 3.4.3 (3) and by Example 3.4.2.

Set $r = -\tau$. Note that, under the conditions $f_1, f_2 \in V_r \cap \mathscr{V}_r$ and $r_1, r_2 \geqslant 0$, the products are well defined. Therefore, $\varphi_1 * \varphi_2 \in V_r(f_1 * f_2)$. □

6.11 Shift-invariant spaces and wave fronts

Using the set of wave fronts, the elements of the spaces $\mathscr{P}'$ and V_r are described in the following assertions.

Theorem 6.11.1 ([8]). *Let $\Gamma \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ be an open convex cone. If*

$$v = \sum_{q\in\mathbb{Z}^n} \alpha_q \, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \in \mathscr{P}', \qquad \sum_{q\in\Gamma\cap\mathbb{Z}^n} |\alpha_q|^2 \mu_{2r}(q) < +\infty,$$

then $(x_0,t_0) \notin WF_r(v)$ for every $(x_0,t_0) \in \mathbb{R}^n \times \Gamma$.

Proof. Let $\phi \in \mathscr{D}(\mathbb{T}^n_{x_0,\theta})$ so that $\phi = \mathbf{1}$ in $\mathbb{T}^n_{x_0,\varepsilon}$, where $0<\varepsilon<\theta$. It is known that $\widehat{\phi} \in \mathscr{S}$, by (2.3.7) and Theorem 3.3.1 (3). Choose $\Gamma_{t_0} \subset \Gamma$ and $\Gamma_1 \subset\subset \Gamma_{t_0}$, i.e $\Gamma_1 \cap \mathbb{S}^{n-1}$ is a compact subset of $\Gamma_{t_0} \cap \mathbb{S}^{n-1}$. Then, there is $C>0$ so that

$$t \in \Gamma_1 \quad \wedge \quad q \in \mathbb{Z}^n \cap \big((\mathbb{R}^n \setminus \{\mathbf{0}\}) \setminus \Gamma_{t_0}\big) \quad \Rightarrow \quad \mu_1(t-q) \geqslant C\mu_1(q), \qquad (6.11.1)$$

by elementary geometry. Using Peetre's inequality (3.5.5) (with $x=q$ and $2r$ instead of r), by Theorem 3.2.3 (2) and Example 3.4.2, it follows that

$$\begin{aligned}
\int_{\Gamma_1} |\widehat{\phi v}(t)|^2 \mu_{2r}(t)\,\mathrm{d}t &= \int_{\Gamma_1} |(\widehat{\phi} * \widehat{v})(t)|^2 \mu_{2r}(t)\,\mathrm{d}t \\
&= \int_{\Gamma_1} \Big|\Big(\widehat{\phi} * \sum_{q\in\mathbb{Z}^n} \alpha_q \delta_q\Big)(t)\Big|^2 \mu_{2r}(t)\,\mathrm{d}t \\
&= \int_{\Gamma_1} \Big|\sum_{q\in\mathbb{Z}^n} \alpha_q T_q\widehat{\phi}(t)\Big|^2 \mu_{2r}(t)\,\mathrm{d}t \\
&\leqslant \int_{\Gamma_1} \Bigg(\sum_{q\in\mathbb{Z}^n} |\alpha_q||T_q\widehat{\phi}(t)|^{\frac12}|T_q\widehat{\phi}(t)|^{\frac12}\Bigg)^2 \mu_{2r}(t)\,\mathrm{d}t \\
&\leqslant \int_{\Gamma_1} \Bigg(\sum_{q\in\mathbb{Z}^n} |\alpha_q|^2|T_q\widehat{\phi}(t)|\Bigg)\Bigg(\sum_{q\in\mathbb{Z}^n} |T_q\widehat{\phi}(t)|\Bigg) \mu_{2r}(t)\,\mathrm{d}t \\
&\leqslant C\int_{\Gamma_1} \Bigg(\sum_{q\in\mathbb{Z}^n} |\alpha_q|^2|T_q\widehat{\phi}(t)|\mu_{2r}(t)\Bigg)\,\mathrm{d}t \\
&\leqslant C\int_{\Gamma_1} \Bigg(\sum_{q\in\mathbb{Z}^n} |\alpha_q|^2\mu_{2r}(q)|T_q\widehat{\phi}(t)|\mu_{2|r|}(t-q)\Bigg)\,\mathrm{d}t \\
&= C\cdot I,
\end{aligned}$$

using the fact that $\sum_{q\in\mathbb{Z}^n} |T_q\widehat{\phi}(t)| < +\infty$, $t\in\mathbb{R}^n$, since $\widehat{\phi}\in\mathscr{S}$. Further,

$$I = \Bigg(\int_{\Gamma_1} \sum_{q\in\mathbb{Z}^n\cap\Gamma_{t_0}} + \int_{\Gamma_1} \sum_{q\in\mathbb{Z}^n\setminus\Gamma_{t_0}}\Bigg)|\alpha_q|^2\mu_{2r}(q)|T_q\widehat{\phi}(t)|\mu_{2|r|}(t-q)\,\mathrm{d}t = I_1 + I_2.$$

For I_1,

$$I_1 = \sum_{q\in\mathbb{Z}^n\cap\Gamma_{t_0}} |\alpha_q|^2\mu_{2r}(q)\int_{\Gamma_1} |T_q\widehat{\phi}(t)|\mu_{2|r|}(t-q)\,\mathrm{d}t \leqslant C \sum_{q\in\mathbb{Z}^n\cap\Gamma_{t_0}} |\alpha_q|^2\mu_{2r}(q) < +\infty,$$

since $\int_{\mathbb{R}^n} |T_q\widehat{\phi}(t)|\mu_{2|r|}(t-q)\,\mathrm{d}t < +\infty$, $q \in \mathbb{Z}^n$, because $\widehat{\phi} \in \mathscr{S}$. Next, note that $v \in \mathscr{P}'$ implies that $\sum_{q\in\mathbb{Z}^n} |\alpha_q|^2\mu_{-2\tau}(q) < +\infty$ for some $\tau > 0$. Using (6.11.1), it follows that

$$\frac{\mu_{2(\tau+r)}(q)}{\mu_{2(\tau+r)}(t-q)} \leqslant C, \quad t \in \Gamma_1,\ q \in \mathbb{Z}^n \setminus \Gamma_{t_0}.$$

Therefore,

$$\begin{aligned} I_2 &= \int_{\Gamma_1} \sum_{q\in\mathbb{Z}^n\setminus\Gamma_{t_0}} |\alpha_q|^2\mu_{-2\tau}(q)\frac{\mu_{2(\tau+r)}(q)}{\mu_{2(\tau+r)}(t-q)}\mu_{2(\tau+r+|r|)}(t-q)|T_q\widehat{\phi}(t)|\,\mathrm{d}t \\ &\leqslant C \sum_{q\in\mathbb{Z}^n\setminus\Gamma_{t_0}} |\alpha_q|^2\mu_{-2\tau}(q)\int_{\Gamma_1} \mu_{2(\tau+r+|r|)}(t-q)|T_q\widehat{\phi}(t)|\,\mathrm{d}t < +\infty, \end{aligned}$$

since $\int_{\mathbb{R}^n} \mu_{2(\tau+r+|r|)}(t-q)|T_q\widehat{\phi}(t)|\,\mathrm{d}t < +\infty$, $q \in \mathbb{Z}^n$, $\tau > 0$, because $\widehat{\phi} \in \mathscr{S}$.

Hence, the statement follows. □

Corollary 6.11.1 ([8]). *Let $f \in \mathscr{D}$ and $\varphi \in V_r(f)$ such that*

$$\varphi = \sum_{q\in\mathbb{Z}^n} \alpha_q T_q f \quad \text{and} \quad \sum_{q\in\mathbb{Z}^n\cap\Gamma} |\alpha_q|^2\mu_{2r}(q) < +\infty$$

for an open cone $\Gamma \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$. If $\sum_{q\in\mathbb{Z}^n} |\alpha_q|^2\mu_{-2\tau}(q) < +\infty$ for some $\tau > 0$, then for every $(x,t) \in \mathbb{R}^n \times \Gamma$, $(x,t) \notin WF_r(\widehat{\varphi})$.

Proof. Obviously, $\widehat{\varphi} = \widehat{f}\varphi_0$, where $\varphi_0 = \sum_{q\in\mathbb{Z}^n} \alpha_q\,\mathrm{e}^{-2\pi i\langle q,\cdot\rangle}$. Applying Theorem 6.11.1, it follows that for every $(x,t) \in \mathbb{R}^n \times \Gamma$, $(x,t) \notin WF_r(\varphi_0)$. Finally, by Lemma 4.5.3 (1), $(x,t) \notin WF_r(\widehat{\varphi})$. □

In order to determine the conditions for the existence of the product of elements from SI spaces by the set of wave fronts, and the conditions for belonging of that product to some SI space, it is necessary to first investigate the product of elements of the space $\mathscr{P}'$.

The following consideration of sets Λ_1 and Λ_2 is especially interesting. Choose the cones Γ_1 and Γ_2 so that $pr_2\big(WF_{r_1}(f_1)\big) \subset \Gamma_1$, $pr_2\big(WF_{r_2}(f_2)\big) \subset \Gamma_2$, and set

$$\Lambda_1 = \mathbb{Z}^n \cap \Gamma_1 \quad \text{and} \quad \Lambda_2 = \mathbb{Z}^n \cap \Gamma_2.$$

Theorem 6.11.2 ([8]). *Let $v_1, v_2 \in \mathscr{P}'$ (i.e. $v_1 \in \mathscr{P}'^{\tau_1}$, $v_2 \in \mathscr{P}'^{\tau_2}$), $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ be cones such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$ and let the following conditions hold.*

1. *There are $C > 0$ and $a \geqslant 1$ so that*

$$\operatorname{card}\{p \in \mathbb{Z}^n \,:\, q-p \in \Gamma_1 \ \wedge\ p \in \Gamma_2\} \leqslant C|q|^a, \quad q \in \mathbb{Z}^n.$$

2. *Let $(x_0,t_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$ and let $\phi \in \mathscr{D}(\mathbb{T}^n_{x_0,\theta})$, $\phi = \mathbf{1}$ in $\mathbb{T}^n_{x_0,\varepsilon}$, $0 < \varepsilon < \theta$, be such that*

$$pr_2\big(WF_{r_1}(\phi v_1)\big) \subset \Gamma_1 \quad \text{and} \quad pr_2\big(WF_{r_2}(\phi v_2)\big) \subset \Gamma_2,$$

   where $r_1 \geqslant \tau_2$ and $r_2 \geqslant \tau_1$.

Then, $v = (\phi v_1)_{pe}(\phi v_2)_{pe}$ exists in $\mathscr{D}'$. Furthermore, $v \in \mathscr{P}'$.

Proof. Let

$$(\phi v_1)_{pe} = \sum_{q\in\mathbb{Z}^n} \alpha_{q,1}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle} \quad \text{and} \quad (\phi v_2)_{pe} = \sum_{q\in\mathbb{Z}^n} \alpha_{q,2}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}\,.$$

Note, if $x \in \operatorname{supp}\phi$ and $t \in (\mathbb{R}^n \setminus \{\mathbf{0}\}) \setminus \Gamma_1$, then $(x,t) \notin WF_{r_1}(\phi v_1)$. The same holds for ϕv_2. Since $v_1 \in \mathscr{P}'^{\tau_1}$ and $v_2 \in \mathscr{P}'^{\tau_2}$, it follows that

$$\sum_{q\in\mathbb{Z}^n\cap\Gamma_1} |\alpha_{q,1}|^2 \mu_{-2\tau_1}(q) < +\infty \quad \text{and} \quad \sum_{q\in\mathbb{Z}^n\cap\Gamma_2} |\alpha_{q,2}|^2 \mu_{-2\tau_2}(q) < +\infty.$$

Now, using the same procedure as in the proof of Theorem 6.10.1, it is shown that $v = (\phi v_1)_{pe}(\phi v_2)_{pe}$ exists and $v \in \mathscr{P}'$. ☐

Remark 6.11.1. *Theorem* 6.11.2 *can be generalized to the case when there are several cones* Γ_1^k, $k = 1,\ldots,s_1$ *(connected with* v_1*) and* Γ_2^l, $l = 1,\ldots,s_2$ *(connected with* v_2*) such that* $\mathbb{Z}^n \cap \Gamma_1^k$, $k = 1,\ldots,s_1$, *and* $\mathbb{Z}^n \cap \Gamma_2^l$, $l = 1,\ldots,s_2$, *contain index sets for* v_1 *and* v_2, *which are compatible index sets.*

Remark 6.11.2. *In the case* $n = 2$ *if* $\Gamma_1 \cap (-\Gamma_2) = \emptyset$, *then it follows that condition* (1) *of Theorem* 6.11.2 *holds with* $a = 2$ *(hypothesis is that condition* (1) *also holds for* $n > 2$ *with* $a = n$, *but the structure of cones and their intersections are more complex in the dimension* $n > 2$, *thus this is an open problem for now).*

Indeed, suppose that cones are acute (if not, they can be divided into finite sets of cones). Thus, let Γ_1 *and* $-\Gamma_2$ *be acute and let* $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. *By translation the cone* $-\Gamma_2$ *for vector* $\overrightarrow{OQ}$, *where* $O = (0,0)$ *and* $Q = (q_1,q_2)$, *there are several different positions of the cone. Cones may not have an intersection, but the most interesting case is when they intersect in four points (in that case, the intersection has the largest number of possible integer points, that is, the surface of the intersection is the largest). Therefore (without losing generality), let*

$$\Gamma_1 = \{(x_1,x_2) : kx_1 \geqslant x_2,\ x_1 \geqslant 0\} \quad \text{and} \quad -\Gamma_2 = \{(x_1,x_2) : lx_1 \geqslant x_2,\ x_1 \leqslant 0\},$$

where $l > k > 0$ *(see Figure* 3*).*

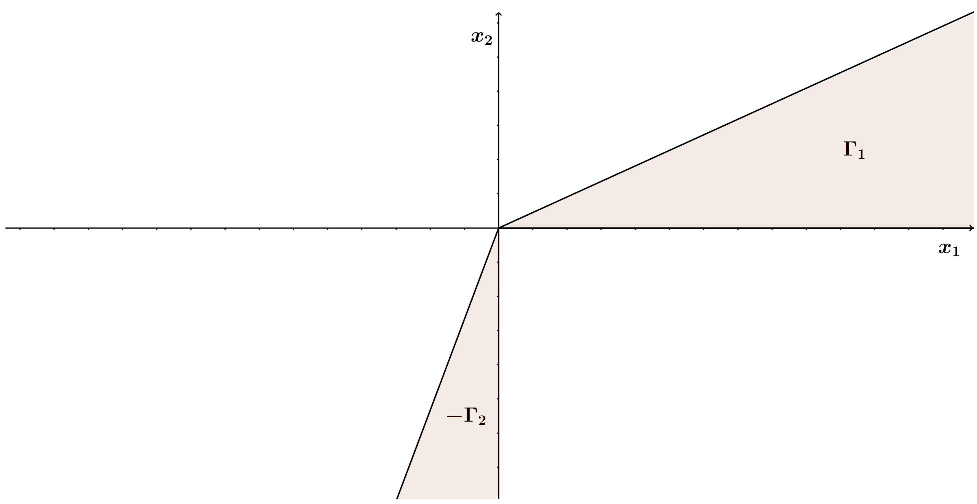

Figure 3.

Let the cone $-\Gamma_2$ be in the position shown in Figure 4, after the translation for the vector $\overrightarrow{OQ}$. The points of intersection are denoted by A_1, A_2, A_3, A_4. Each point of intersection can be written in the form

$$(c^j_{1,1}q_1 + c^j_{1,2}q_2, c^j_{2,1}q_1 + c^j_{2,2}q_2), \quad j = 1, 2, 3, 4.$$

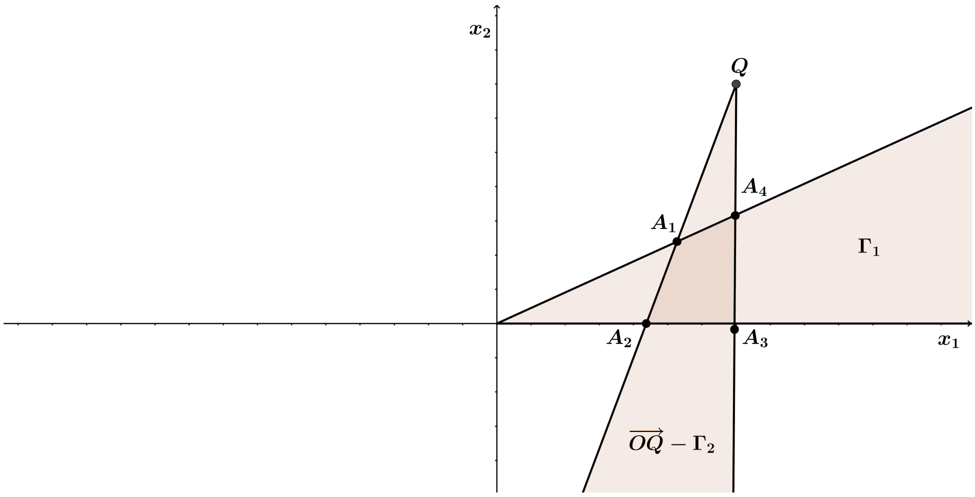

Figure 4.

Now, by calculating the obtained area of the intersection, it is concluded that it can be estimated by $C(q_1^2 + q_2^2)$, i.e by $C|q|^2$ for some $C > 0$. Hence, in the dimension $n = 2$, the assumption $\Gamma_1 \cap (-\Gamma_2) = \emptyset$ implies the condition (1) of Theorem 6.11.2 with $a = 2$.

The immediate consequences of the theorems 6.10.2 and 6.11.2 are given in the following assertions.

Corollary 6.11.2 ([8])**.** *Let $f_k \in H^{r_k}$, $k = 1, 2$, and let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n \setminus \{\mathbf{0}\}$ be cones such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Suppose that the assumption (1) of Theorem 6.11.2 holds.*

(1) *Let $x_0 \in \mathbb{R}^n$, $\varphi_1, \varphi_2 \in \mathscr{D}'$ and $\phi \in \mathscr{D}(\mathbb{T}^n_{x_0,\theta})$ such that $\phi = \mathbf{1}$ in $\mathbb{T}^n_{x_0,\varepsilon}$, $0 < \varepsilon < \theta$. Suppose that*

$$\mathscr{F}[\phi\varphi_1] = \widehat{f}_1 \sum_{q\in\mathbb{Z}^n} \alpha_{q,1}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}, \qquad \mathscr{F}[\phi\varphi_2] = \widehat{f}_2 \sum_{q\in\mathbb{Z}^n} \alpha_{q,2}\, \mathrm{e}^{-2\pi i\langle q,\cdot\rangle}$$

and for some $\tau_1, \tau_2 > 0$ hold

$$\sum_{q\in\mathbb{Z}^n\cap\Gamma_1} |\alpha_{q,1}|^2 \mu_{-2\tau_1}(q) < +\infty \quad \text{and} \quad \sum_{q\in\mathbb{Z}^n\cap\Gamma_2} |\alpha_{q,2}|^2 \mu_{-2\tau_2}(q) < +\infty.$$

Moreover, suppose that the condition (2) of Theorem 6.11.2 holds. Then, there is an $r \in \mathbb{R}$ such that

$$\phi\varphi_1 = \sum_{q\in\mathbb{Z}^n} \alpha_{q,1} T_q f_1 \quad \text{and} \quad \phi\varphi_2 = \sum_{q\in\mathbb{Z}^n} \alpha_{q,2} T_q f_2$$

are elements of $V_r(f_1)$ and $V_r(f_2)$, respectively, and $(\phi\varphi_1) * (\phi\varphi_2) \in V_r(f_1 * f_2)$.

(2) *Let* $\varphi_k \in V_{r_k}(f_k)$, $k = 1, 2$, *and let* $(x_0, t_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{\mathbf{0}\})$. *For* $\widehat{\varphi_k} = \widehat{f}_k v_k$ *suppose that* $pr_2\big(WF_{r_k}(v_k)\big) \subset \Gamma_k$, *where* $v_k = \sum_{q \in \mathbb{Z}^n} \alpha_{q,k} \, \mathrm{e}^{-2\pi i \langle q, \cdot \rangle}$, $k = 1, 2$. *Moreover, suppose that hold*

$$\sum_{q \in \mathbb{Z}^n \cap \Gamma_1} |\alpha_{q,1}|^2 \mu_{-2\tau_1}(q) < +\infty \quad \textit{and} \quad \sum_{q \in \mathbb{Z}^n \cap \Gamma_2} |\alpha_{q,2}|^2 \mu_{-2\tau_2}(q) < +\infty,$$

where $r_1 \geqslant \tau_2 > 0$ *and* $r_2 \geqslant \tau_1 > 0$. *Then, there is an* $r \in \mathbb{R}$ *such that*

$$\varphi = \varphi_1 * \varphi_2 \in V_r(f_1 * f_2) \quad \textit{and} \quad \varphi = \sum_{q \in \mathbb{Z}^n} \alpha_q T_q(f_1 * f_2).$$

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Aleksandar Aksentijević was born on February 11th, 1991 in Jagodina. Until the second semester of the fourth grade, he attended Primary School "Ljubiša Urošević" in Ribare near Jagodina, and then Primary School "Vuk Karadžić" in Ćuprija. He completed Primary School in 2006 as an excellent student. He graduated from the Technical School in Ćuprija, majoring in Mechanical Engineering for Computer Engineering, in 2010 as the student of the generation. In the same year, he enrolled undergraduate academic studies in Mathematics at the Faculty of Science, University of Kragujevac (FSUKG). He completed his undergraduate studies, Department of Theoretical Mathematics in 2014 with the GPA 9.49. He enrolled Master studies at FSUKG, Department of Theoretical Mathematics, in 2014 and completed them in 2015 with the GPA 10.00, as the best graduate student. In 2016 he enrolled doctoral academic studies in Mathematics, Department of Analysis, at the Doctoral School of Mathematics within FSUKG. He spent the spring semester of the 2019/2020 academic year at Linnaeus University in Växjö, Sweden, through Erasmus+ student exchange.

Aleksandar worked at FSUKG first as a Teaching Associate from 2015 to 2017, and then as a Teaching Assistant from 2017 to 2023. He worked at the Faculty of Technical Sciences in Čačak as an Associate in higher education in the summer semester of 2023. Since December 2023 he has been employed as a Junior Research Associate at FSUKG on the PRISMA program, project GOALS, #2727.

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Publications:

(1) **A. Aksentijević**, S. Aleksić and S. Pilipović, *The structure of shift-invariant subspaces of Sobolev spaces*, Theor Math Phys 218, 177–191, 2024.

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