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CALCULATION OF FLOW PAST A SPHERE NEAR THE GROUND USING AN INDIRECT BOUNDARY ELEMENT METHOD

G. Muhammad, N. A. Shah and M. Mushtaq

Department of Mathematics
University of Engineering & Technology, Lahore – 54890 *Pakistan*
e-mail: chgm2004@yahoo.com

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ABSTRACT. In this paper, indirect boundary element method (IBEM) is applied to calculate flow past a sphere near the ground. Indirect boundary element method (IBEM) is one of the types of boundary element method (BEM) formulations. It is based on the singularities such as sources or doublets over the body surface and computes unknowns in the form of singularity strength. This method is more general and flexible for the solution of a given problem. It is the most popular method due its accuracy and efficiency and it needs less computational labor as well computational costs i.e. it is time saving and economical, which establishes its superiority over other domain types of methods such as finite element method (FEM), finite difference method (FDM) and finite volume method (FDM), etc. The accuracy of computed results can be increased by increasing the number of boundary elements. The validity of this method is well checked by given table and graphs.

Keywords: Indirect boundary element method, Inviscid flow, Axisymmetric flow, Flow past, Sphere, ground.

INTRODUCTION

In present period of science and technology, the popularity of boundary element methods (BEMs) rises for solving fluid flow problems and modeling physics in fluids, Which provide the best base for the numerical methodology to solve the fluid flow problems and they provide the best solution of boundary integral equations (BIEs) based on a discretization process .The applications of BEMs rose on sound footing popular with the invention of electronic computers. The boundary element methods attained maturity in the late 1970s. The term “boundary elements” opened eyes in the department of civil engineering at Southampton University, United Kingdom in 1977. Full-fledged research in boundary element methods reached its peak in 1990. In past, these methods were well known under different names such as ‘panel methods’, ‘surface singularity methods’, ‘boundary integral equation methods’ or ‘boundary integral solutions’. First of all, finite difference method (FDM) and finite element method (FEM) and finite volume method (FVM),etc were being used to solve numerically the problems in computational fluid dynamics (CFD). But later on, the boundary element methods gained popularity over domain type methods due to their advantages. One of the advantages is that with boundary element methods one has to define the whole body surface, whereas with domain methods it is essential to discretize the entire flow field. The most important characteristics of BEMs are the much smaller system of equations and considerable reduction in data, which are perquisite to run a computer program efficiently. So one can say that BEMs are time-saving, efficient and accurate. Furthermore, boundary element methods are well suited to flow problems with infinite domains. These methods are classified into 'direct' and 'indirect' methods. The direct method takes the form of a statement, which provides the values of unknown variables at any flow field point in terms of the complete set of all the boundary data. The indirect method utilizes a distribution of singularities over the surface of the body and computes this distribution as the solution of integral equation. The equation for the indirect method can be obtained from the equation for the direct method and can also be interpreted as a weighted residual formulation. Indirect boundary element method is essential the method for solving partial differential equations arising in problems in such diverse topics as stress analysis, heat transfer and electromagnetic theory, potential theory, fracture mechanics, fluid mechanics, elasticity, elastostatics and elastodynamics , etc. This method is now being used for the solutions of incompressible flows around complex configurations. Thus it can be said that Indirect boundary element method is powerful numerical technique receiving much attention from researchers, engineering community and is offering the numerical solutions of a large number of physical problems of different types. In IBEM, the integral equation is approximated on the boundary of flow field so that the creation of meshes is only required on the boundary in the two-dimensional space, although the fluid flow problems are in three-dimensional space. Thus the computational cost and time in this method is much smaller than other numerical methods due to very small number of meshes. That is why the IBEM is more attractive among the computational researchers and specialists in dealing with infinite regions. Finally, it is more simple, flexible and economical than other numerical methods.

FLOW PAST A SPHERE:

Suppose that a sphere of unit radius with center at origin is near the ground in a uniform stream of velocity U in the +ve direction of z–axis as shown in fig. 1.

The inviscid flow over the ground, which is considered to be a plane, remains everywhere tangential to its surface. This flow is simulated by the so–called 'mirror image' principle. A mirror image of the sphere is imagined to be present below the ground plane. The flow field by a uniform stream parallel to the ground plane is disturbed by the sphere and its image is therefore symmetrical with respect to the ground plane. The plane of symmetry is a stream surface and represents the ground. An axisymmetric flow is most conveniently formulated in cylindrical polar coordinates. The cylindrical polar coordinates are taken as (r, θ, z).

The velocity potential and stream function for a sphere of radius a moving in the -ve direction of z–axis with velocity U can be calculated as

$$\phi = -\frac{1}{2} U \frac{a^3}{r^2} \cos\theta , \quad \psi = \frac{1}{2} U \frac{a^3}{r^2} \tag{1}$$

Also the velocity potential and stream function for a uniform stream moving with velocity U in the +ve direction of z–axis are given by

$$\phi = -U r \cos\theta , \quad \psi = -\frac{1}{2} U r^2 \sin^2\theta \tag{2}$$

Therefore the velocity potential and stream function for the streaming motion past a fixed sphere in the +ve direction of z–axis take the forms

$$\phi \;=\; -\,U\,r\cos\theta - \frac{1}{2}\,U\,\frac{a^3}{r^2}\cos\theta \;=\; -\,U\left(r + \frac{a^3}{2\,r^2}\right)\cos\theta \qquad (3)$$

and $$\psi \;=\; -\frac{1}{2}\,U\,r^2\sin\theta + \frac{1}{2}\,U\,\frac{a^3}{r}\sin^2\theta \;=\; -\frac{1}{2}\,U\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta \qquad (4)$$

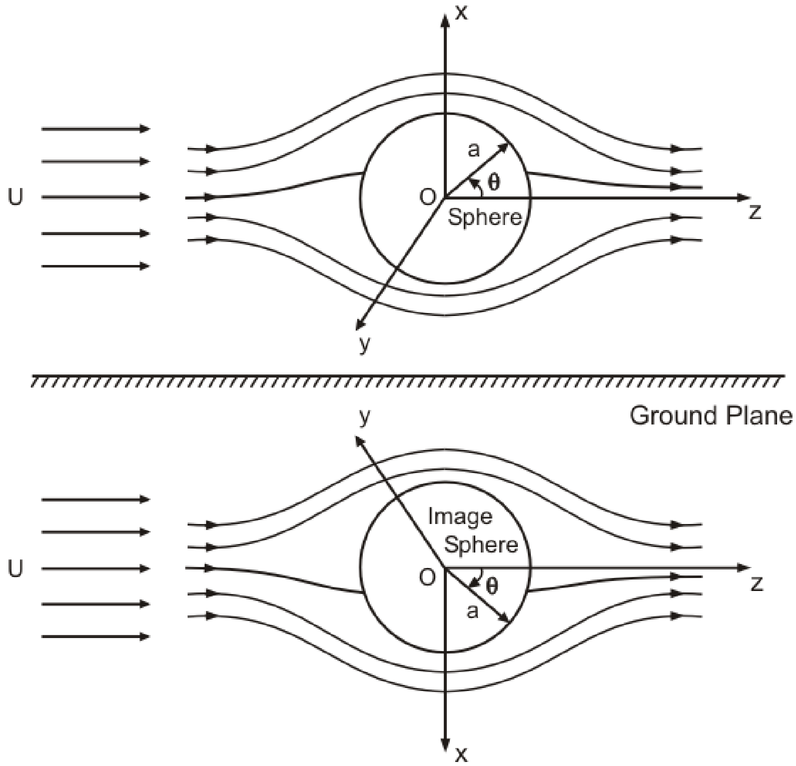


Fig. 1. – Streaming past a sphere near the ground

The velocity components at any point (r, θ) are given by

$$v_r \;=\; -\frac{\partial\phi}{\partial r} \;=\; U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$v_\theta \;=\; -\frac{1}{r}\,\frac{\partial\phi}{\partial r} \;=\; -\,U\left(1 + \frac{a^3}{2\,r^3}\right)\sin\theta$$

$$v_r \;=\; -\frac{1}{r^2\sin\theta}\,\frac{\partial\psi}{\partial\theta} \;=\; -\frac{1}{r^2\sin\theta}\left[-\,U\left(r^2 - \frac{a^2}{r}\right)\sin\theta\cos\theta\right]$$

$$=\; U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$v_\theta \;=\; \frac{1}{r\sin\theta}\,\frac{\partial\psi}{\partial r} \;=\; \frac{1}{r\sin\theta}\left[-\frac{1}{2}\,U\left(2\,r + \frac{a^3}{r^2}\right)\sin^2\theta\right]$$

$$=\; -\,U\left(1 + \frac{a^3}{2\,r^3}\right)\sin\theta$$

The speed at any point in the flow field is given by

$$v = \sqrt{v_r^2 + v_\theta^2}$$

$$= \sqrt{\left[U\left(1 - \frac{a^3}{r^3}\right)\sin\theta \right]^2 + \left[-U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \right]^2}$$

Therefore the speed any point on the sphere itself is given by

$$v = \sqrt{0 + U^2\left(1 + \frac{1}{2}\right)^2 \sin^2\theta}$$

$$= \sqrt{\frac{9}{4}U^2 \sin^2\theta}$$

$$= \frac{3}{2}U\sin\theta \qquad (5)$$

Now the pressure distribution at any point of the flow field can be calculated by using the Bernoulli's equation between two points i.e.

$$\frac{p}{\rho} + \frac{1}{2}v^2 = \frac{p_\infty}{\rho} + \frac{1}{2}U^2$$

or $$p = p_\infty + \frac{1}{2}\rho\,(U^2 - V^2) \qquad (6)$$

where p_∞ is the pressure at infinity

Equation (6) takes the form while using equation (5) .

$$p = p_\infty + \frac{1}{2}\rho\,U^2\left(1 - \frac{9}{4}\sin^2\theta\right) \qquad (7)$$

Now the pressure is maximum and minimum at the points where $\theta = 0$ or π and $\theta = \pm \frac{\pi}{2}$ respectively

Thus $p_{max} = p_\infty + \frac{1}{2}\rho\,U^2$ and $p_{min} = p_\infty - \frac{5}{8}\rho\,U^2$

Therefore from equation (7), the pressure coefficient C_p on the boundary of the sphere is given by

$$C_p = 1 - \frac{9}{4}\sin^2\theta \qquad (8)$$

From equations (5) and (8)

$C_p = 1 - V^2$ taking $U = 1$

BOUNDARY CONDITIONS

The boundary conditions to be satisfied over the surface of a sphere is

$$\frac{\partial \phi_{sphere}}{\partial n} = U(\hat{n} \cdot \hat{k}) = U \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (9)$$

where ϕ_{sphere} is the perturbation velocity potential and $\hat{n}$ is the unit normal draw outward from the surface of the sphere.

The equation of the surface of a sphere is $x^2 + y^2 + z^2 = 1$

where the radius a of the sphere is taken to be 1.

Thus equation (9) becomes

$$\frac{\partial \phi_{sphere}}{\partial n} = U z = z, \text{ taking } U = 1$$

MATHEMATICAL FORMULATION OF INDIRECT BOUNDARY ELEMENT METHOD

For exterior flow for three-dimensional problems, the mathematical formulation for indirect boundary element method in terms of doublets distribution over the boundary S of the body is given by

$$-\frac{1}{2}\Phi_i + \phi_\infty + \iint_{S-i} \Phi \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS = z_i \qquad (10)$$

Which is discretized by dividing the boundary of the body under consideration into ‘m’ elements and finally, it is written in matrix form as $[H]\{\underline{U}\} = \{\underline{R}\}$ (11)

Whereas usual [H] is a matrix of influence coefficients, $\{\underline{U}\}$ is a vector of unknown total potentials Φ_p and $\{\underline{R}\}$ on the R.H.S. is a known vector whose elements are the negative of the values of the velocity potential of the uniform stream at the nodes on the boundary of the body.

METHOD OF ELEMENT DISTIBUTIONS:

The indirect boundary element method is used for calculating the potential flow solution around a sphere when such sphere is lying very near to the ground for which the analytical solution is available.

Suppose the side (1) in fig. 2 is that at which the values of the potential are to be evaluated at the fixed points ‘i’. At a given point, the components of

$$\hat{H}_{ij} = \iint_{S_j - i} \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS \quad \text{and} \quad G_{ij} = \iint_{S_j} \left(\frac{1}{4\pi r}\right) dS$$

integrals due to an element on side (1) are evaluated first. The y–coordinates of all nodes of this element are then changed [side (2)] and the components from this reflected element are calculated. The x–coordinates of all nodes of this element are then changed [side (3)] and the components from this element are evaluated. Position (4) is then reached by changing the y–coordinates again and evaluating the components due to the element. The integral components from the corresponding elements on all the four sides are then summed to calculate the total integral values at the point on side (1) due to one particular element on side (1). The process is repeated for all the elements on side (1) and then for all the fixed points. The pressure distribution over the surface of a sphere of radius ‘1’ unit with ground clearance of 0.1 units has been calculated using the above method. The pressure coefficient over the surface of the sphere is calculated for 96 and 384 boundary element.

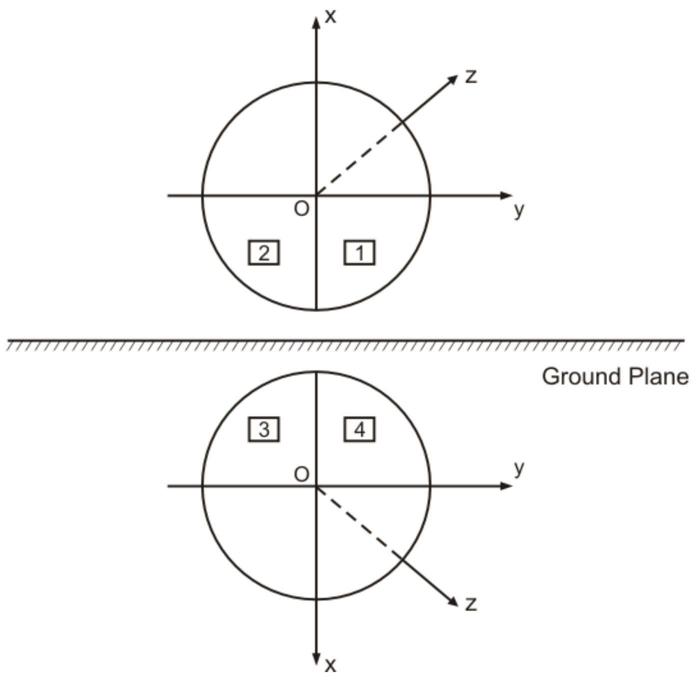


Fig. 2 – Sphere and its image.

For the accuracy of calculated results, such results are compared with the analytical results in N.A. SHAH [7]. The analytical solutions are based upon a truncated series of images of doublets and line doublets. The pressure distribution over the sphere surface can be obtained for 384 to get more accuracy in comparison.

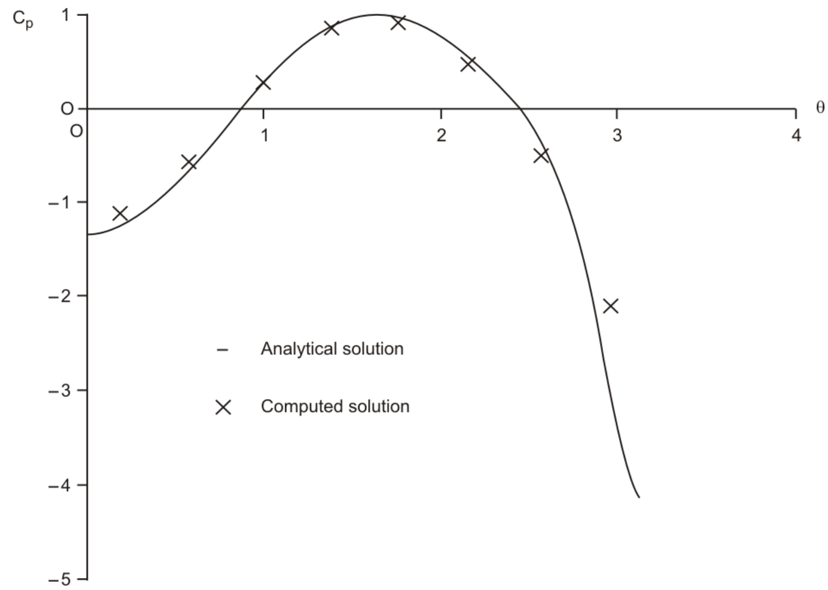
The table of 96 boundary elements for ground clearance of 0.1 units is only given, but the table of 384 boundary elements is large in size and so they are not given in this research paper.

CONCLUSION

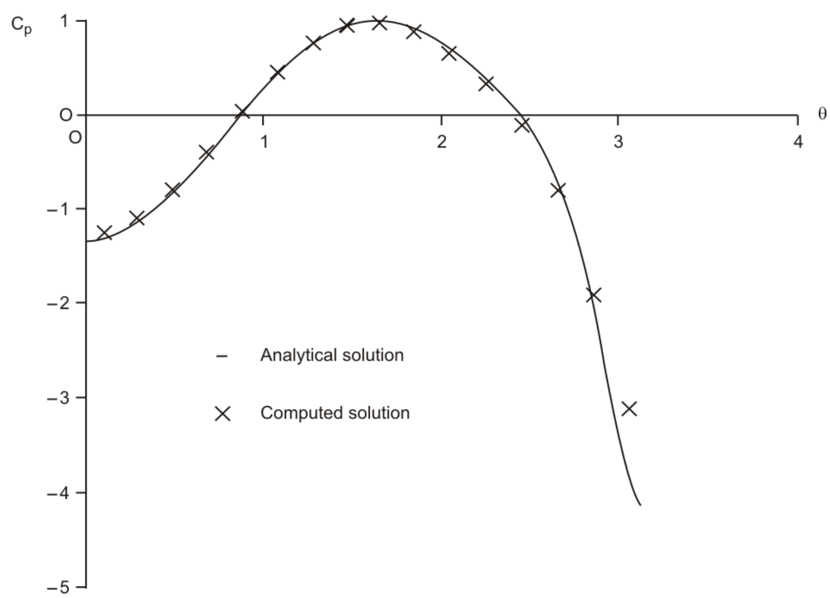
In the present paper, the flow past around a sphere near the ground is calculated by using indirect boundary element method (IBEM). The calculated results obtained by such method for flow past a sphere in the vicinity of ground are excellent in accuracy as shown in the graphs given above. These graphs show that the calculated results are very near to those results in analytical form at half or slightly more points as shown in graphs and they also show that these results are good in agreement with analytical results near the line joining the centers of the sphere and image sphere.

Table 1. – Comparison of analytical and computed results for pressure coefficients over the boundary of a sphere with ground clearance 0.1 units for 96 boundary elements.

ELEMENT	XM	YM	ZM	VELOCITY	CP
1	-.177E+00	-.934E+00	.177E+00	.17902E+01	-.22047E+01
2	-.522E+00	-.798E+00	.157E+00	.12423E+01	-.54331E+00
3	-.798E+00	-.522E+00	.157E+00	.73647E+00	.45761E+00
4	-.934E+00	-.177E+00	.177E+00	.31007E+00	.90386E+00
5	-.934E+00	.177E+00	.177E+00	.38985E+00	.84802E+00
6	-.798E+00	.522E+00	.157E+00	.85698E+00	.26558E+00
7	-.522E+00	.798E+00	.157E+00	.12659E+01	-.60244E+00
8	-.177E+00	.934E+00	.177E+00	.14730E+01	-.11697E+01
9	.177E+00	.934E+00	.177E+00	.14730E+01	-.11697E+01
10	.522E+00	.798E+00	.157E+00	.12659E+01	-.60244E+00
11	.798E+00	.522E+00	.157E+00	.85698E+00	.26558E+00
12	.934E+00	.177E+00	.177E+00	.38985E+00	.84802E+00
13	.934E+00	-.177E+00	.177E+00	.31007E+00	.90386E+00
14	.798E+00	-.522E+00	.157E+00	.73647E+00	.45761E+00
15	.522E+00	-.798E+00	.157E+00	.12423E+01	-.54331E+00
16	.177E+00	-.934E+00	.177E+00	.17902E+01	-.22047E+01
17	-.157E+00	-.798E+00	.522E+00	.16861E+01	-.18428E+01
18	-.470E+00	-.703E+00	.470E+00	.13655E+01	-.86456E+00
19	-.703E+00	-.470E+00	.470E+00	.96450E+00	.69745E-01
20	-.798E+00	-.157E+00	.522E+00	.84561E+00	.28494E+00
21	-.798E+00	.157E+00	.522E+00	.85851E+00	.26297E+00
22	-.703E+00	.470E+00	.470E+00	.99775E+00	.44977E-02
23	-.470E+00	.703E+00	.470E+00	.13292E+01	-.76690E+00
24	-.157E+00	.798E+00	.522E+00	.14721E+01	-.11671E+01
25	.157E+00	.798E+00	.522E+00	.14721E+01	-.11671E+01
26	.470E+00	.703E+00	.470E+00	.13292E+01	-.76690E+00
27	.703E+00	.470E+00	.470E+00	.99775E+00	.44953E-02
28	.798E+00	.157E+00	.522E+00	.85851E+00	.26296E+00
29	.798E+00	-.157E+00	.522E+00	.84561E+00	.28494E+00
30	.703E+00	-.470E+00	.470E+00	.96450E+00	.69742E-01
31	.470E+00	-.703E+00	.470E+00	.13655E+01	-.86456E+00
32	.157E+00	-.798E+00	.522E+00	.16861E+01	-.18428E+01
33	-.157E+00	-.522E+00	.798E+00	.15703E+01	-.14658E+01
34	-.470E+00	-.470E+00	.703E+00	.13722E+01	-.88285E+00
35	-.522E+00	-.157E+00	.798E+00	.12917E+01	-.66853E+00
36	-.522E+00	.157E+00	.798E+00	.12802E+01	-.63880E+00
37	-.470E+00	.470E+00	.703E+00	.13344E+01	-.78056E+00
38	-.157E+00	.522E+00	.798E+00	.14788E+01	-.11867E+01
39	.157E+00	.522E+00	.798E+00	.14788E+01	-.11867E+01
40	.470E+00	.470E+00	.703E+00	.13344E+01	-.78056E+00
41	.522E+00	.157E+00	.798E+00	.12802E+01	-.63880E+00
42	.522E+00	-.157E+00	.798E+00	.12917E+01	-.66853E+00
43	.470E+00	-.470E+00	.703E+00	.13722E+01	-.88285E+00
44	.157E+00	-.522E+00	.798E+00	.15703E+01	-.14658E+01
45	-.177E+00	-.177E+00	.934E+00	.15165E+01	-.12999E+01
46	-.177E+00	.177E+00	.934E+00	.14946E+01	-.12339E+01
47	.177E+00	.177E+00	.934E+00	.14946E+01	-.12339E+01
48	.177E+00	-.177E+00	.934E+00	.15165E+01	-.12999E+01



Graph 1. – Comparison of analytical and computed results for pressure coefficients over the boundary of a sphere with ground clearance 0.1 units for 96 boundary elements.



Graph 2. – Comparison of analytical and computed results for pressure coefficients over the boundary of a sphere with ground clearance 0.1 units for 384 boundary elements.

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