Kragujevac J. Sci. 27 (2005) 67-74.

EQUIENERGETIC COMPLEMENT GRAPHS

Harishchandra S. Ramane[a], Ivan Gutman[b], Hanumappa B. Walikar[c] and Sabeena B. Halkarni[c]

[a]*Department of Mathematics, Gogte Institute of Technology, Udyambag, Belgaum – 590008, India,*
[b]*Faculty of Science, P. O. Box 60, 34000,Kragujevac, Serbia & Montenegro,*
[c]*Department of Mathematics, Karnatak University, Dharwad – 580003, India.*

(Received August 3, 2004)

ABSTRACT. The energy of a graph G is the sum of the absolute values of its eigenvalues. Two graphs are said to be equienergetic if their energies are equal. In this paper we show that if G is a regular graph on n vertices and of degree $r \geq 3$, then $E(\overline{L^2(G)}) = (nr-4)(2r-3)-2$. This leads to the construction of infinitely many equienergetic graphs, which are of the same order and noncospectral.

INTRODUCTION

The concept of graph energy was introduced by one of the present authors [8], motivated by results obtained by applying graph spectral theory to molecular orbital theory [7,14]. For recent mathematical work on the energy of a graph see [1,9,12,819-23,26,29-33] whereas for recent chemical studies see [2,3,5,6,10,11,13,15-17,27,28].

Let G be an undirected graph without loops and multiple edges on n vertices. The eigenvalues of the adjacency matrix of G are said to be the eigenvalues of G and they are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ and are labeled so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. These eigenvalues form the spectrum of G [4]. Two graphs are said to be cospectral if they have the same spectra.

The energy of a graph G is defined as [8], $E(G) = \sum_{i=1}^{n} |\lambda_i|$. Two graphs G_1 and G_2 are said to be equienergetic if $E(G_1) = E(G_2)$. Cospectral graphs are equienergetic. If O_k is the k-vertex graph without edges and G any graph, then G and $G \cup O_k$ are equienergeti. These two trivial cases of equienergeticity are, of course, of no interest. Quite recently classes of non-cospectral equienergetic graphs were designed [1,3,23,26], among which also pairs of equienergetic chemical trees [3]. In this paper we point out further classes of equienergetic graphs.

Let G be a graph and $L^1(G) = L(G)$ be its line graph [18]. Further, let $L^k(G) = L(L^{k-1}(G))$, $k \geq 2$, be the iterated line graphs of G. A graph G is said be regular of degree r if all its vertices have same degree, equal to r. If G is a regular graph on n vertices and of degree r, then $L(G)$ is a regular graph on

$$n_1 = nr/2 \tag{1}$$

vertices and of degree

$$r_1 = 2r - 2. \tag{2}$$

Consequently all iterated line graphs $L^k(G)$ of a regular graph G are regular [18]. In particular, if G is a regular graph on n vertices, of degree r then by Eqs. (1) and (2), $L^2(G)$ is a regular graph on $n_2 = n_1 r_1/2 = nr(r-1)/2$ vertices and of degree $r_2 = 2r_1 - 2 = 4r - 6$. For more details on line graphs see elsewhere [18].

Theorem 1 [4]. If G is a regular graph on n vertices and of degree r, then its largest eigenvalue is $\lambda_1 = r$.

Theorem 2 [25]. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a regular graph G on n vertices and of degree r, then the eigenvalues of $L(G)$ are $\lambda_i + r - 2$, $i = 1, 2, \ldots, n$, and -2, $n(r-2)/2$ times.

Theorem 3 [24]. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a regular graph *G* of order *n* and of degree *r*, then the eigenvalues of $\overline{G}$, the complement of *G*, are $n - r - 1$ and $-\lambda_i - 1$, $i = 2, 3, \ldots, n$.

Theorem 4 [23]. If *G* is a regular graph of order *n* and of degree $r \geq 3$, then

$$E(L^2(G)) = 2nr(r-2). \tag{3}$$

Corollary 5 [23]. Let G_1 and G_2 be two regular graphs, both on *n* vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$ are equienergetic.

EQUIENERGETIC COMPLEMENT GRAPHS

Theorem 6. If *G* is a regular graph of order *n* and of degree $r \geq 3$, then

$$E(\overline{L^2(G)}) = (nr-4)(2r-3)-2. \tag{4}$$

Proof. Let *G* be a regular graph on *n* vertices and of degree $r \geq 3$. Let its eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then by Theorem 2, the eigenvalues of *L*(*G*) are

$$\left.\begin{array}{lll} & \lambda_i + r - 2\,, & i = 1, 2, \ldots, n \\ \text{and} & -2\,, & n(r-2)/2 \text{ times} \end{array}\right\} \tag{5}$$

In view of that fact that *L*(*G*) is a regular graph on *nr*/2 vertices and of degree $2r - 2$, from Eqs. (5) the eigenvalues of $L^2(G)$ are

$$\left.\begin{array}{lll} & \lambda_i + 3r - 6, & i = 1, 2, \ldots, n \\ & 2r - 6, & n(r-2)/ \text{ times} \\ \text{and} & -2, & nr(r-2)/2 \text{ times} \end{array}\right\} \quad (6)$$

Because $L^2(G)$ is a regular graph on *nr*(*r* – 1)/2 vertices and of degree 4*r* – 6, from Theorem 3 and Eqs. (6), the eigenvalues of $\overline{L^2(G)}$ are

$$\left.\begin{array}{lll} & -\lambda_i - 3r + 5, & i = 2, 3, \ldots, n \\ & -2r + 5, & n(r-2)/2 \text{ times} \\ & 1, & nr(r-2)/2 \text{ times} \\ \text{and} & (nr(r-1)/2) - 4r + 5 & \end{array}\right\} \quad (7)$$

If $d_{\max}$ is the greatest vertex degree of a graph, then all its eigenvalues belong to the interval $[-d_{\max}, d_{\max}]$ [4]. In particular the eigenvalues of a regular graph of degree *r*, satisfy the condition $-r \leq \lambda_i \leq r$, *i* = 1,2,…,*n*. If *r* ≥ 3 then $\lambda_i + 3r - 5 > 0$, 2*r* – 5 > 0 and (*nr*(*r* – 1)/ 2) – 4*r* + 5 > 0. Therefore the energy of $\overline{L^2(G)}$ is computed from (7) as

$$E(\overline{L^2(G)}) = \sum_{i=2}^{n} \left| -\lambda_i - 3r + 5 \right| + \left| -2r + 5 \right| \frac{n(r-2)}{2} + \left| 1 \right| \frac{nr(r-2)}{2} + \left| \frac{nr(r-1)}{2} - 4r + 5 \right|$$

$$= \sum_{i=2}^{n} \lambda_i + (3r-5)(n-1) + (2r-5)\frac{n(r-2)}{2} + \frac{nr(r-2)}{2} + \frac{nr(r-1)}{2} - 4r + 5$$

$$= (nr-4)(2r-3) - 2, \qquad \text{since } \sum_{i=2}^{n} \lambda_i = -r.$$

Corollary 7. Let G_1 and G_2 be two regular graphs on *n* vertices and of degree $r \geq 3$. Then $\overline{L^2(G_1)}$ and $\overline{L^2(G_2)}$ are equienergetic.

Proof. Corollary 7 directly follows from Eq. (4).

Corollary 8. Let G_1 and G_2 be two regular graphs on *n* vertices and of degree $r \geq 3$. Then for any $k \geq 2$, $E(\overline{L^k(G_1)}) = E(\overline{L^k(G_2)})$.

Proof. By repeated application of Eqs. (1) and (2), the graphs $L^{k-2}(G_1)$ and $L^{k-2}(G_2)$ have same number of vertices. Because $L^{k-2}(G_1)$ and $L^{k-2}(G_2)$ are regular graphs of same degree, with equal number of vertices, by Corollary 7, $\overline{L^k(G_1)}$ = $\overline{L^2(L^{k-2}(G_1))}$ and $\overline{L^k(G_2)}$ = $\overline{L^2(L^{k-2}(G_2))}$ are equienergetic.

Corollary 9. Let G_1 and G_2 be two non-cospectral regular graphs on *n* vertices, of degree $r \geq 3$. Then for any $k \geq 2$, both $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are regular, non-cospectral, possessing same number of vertices, same number of edges and equienergetic.

Proof. All iterated line graphs $L^k(G)$ of regular graphs are regular and the complement of a regular graph is also regular. Therefore $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are regular graphs. From Eqs. (5), (6), and (7), if G_1 and G_2 are not cospectral then $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are not cospectral, for any $k \geq 1$. By repeated application of Eqs. (1) and (2), we conclude that $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ possess equal number of vertices and from Corollary 8, that $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic.

From Eqs. (3) and (4), we arrive at the following:

Corollary 10. If *G* is a regular graph on *n* vertices and of degree $r \geq 3$, then $E(L^2(G))$ = $E(\overline{L^2(G)}) - r(n-8) - 10$.

Corollary 11. Let G be a regular graph on n vertices and of degree $r \geq 3$. Then $E(L^2(G)) = E(\overline{L^2(G)})$ if and only if $G = K_6$.

Proof. If $G = K_6$, then G is a regular graph on 6 vertices and of degree 5. Then from (3) and (4), $E(L^2(G)) = E(\overline{L^2(G)}) = 180$.

Conversely, assume that $E(L^2(G)) = E(\overline{L^2(G)})$

Then $r(n-8) + 10 = 0$. Bearing in mind that $r \geq 3$, the latter condition is satisfied for $n = 7$, $r = 10$ and $n = 6$, $r = 5$. There is no graph with $n = 7$ and $r = 10$. Hence the case that remains is $n = 6$ and $r = 5$, which is K_6.

REFERENCES

[1] R. Balakrishnan, The energy of a graph, *Lin. Algebra Appl.* **387** (2004) 287-295.

[2] Đ. Baralić, I. Gutman, B. Popović, Solution of the Türker inequality, *Kragujevac J. Sci.* **26** (2004) 13-18.

[3] V. Brankov, D. Stevanović, I. Gutman, Equienergetic chemical trees, *J. Serb. Chem. Soc.* **69** (2004) 549-553.

[4] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[5] H. Fripertinger, I. Gutman, A. Kerber, A. Kohnert, D. Vidović, The energy of a graph and its size dependence. An improved Monte Carlo approach, *Z. Naturforsch*. **56** (2001) 342-346.

[6] A. Graovac, I. Gutman, P. E. John, D. Vidović, I. Vlah, On statistics of graph energy, *Z. Naturforsch.* **56a** (2001) 307-311.

[7] A. Graovac, I. Gutman, N. Trinajstić, T. Živković, Graph theory and molecular orbitals. Application of Sachs theorem, *Theor. Chim. Acta* **26** (1972) 67-78.

[8] I. Gutman, The energy of a graph, *Ber. Math.-Stat. Sekt. Forschungszentrum Graz* **103** (1978) 1 – 22.

[9] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applica- tions*, Springer-Verlag, Berlin, 2001, pp. 196-211.

[10] I. Gutman, N. Cmiljanović, S. Milosavljević, S. Radenković, Effect of non-bonding molecular orbitals on total π-electron energy, *Chem. Phys. Lett.* **383** (2004) 171-175.

[11] I. Gutman, N. Cmiljanović, S. Milosavljević, S. Radenković, Dependence of total π-electron energy on the number of non-bonding molecular orbitals, *Monatsh. Chem.* **135** (2004) 765-772.

[12] I. Gutman and Y. Hou, Bipartite unicyclic graphs with greatest energy, *MATCH Commun. Math. Comput. Chem.* **43** (2001) 17-28.

[13] I. Gutman, D. Stevanović, S. Radenković, S. Milosavljević. N. Cmiljanović, Dependence of total π-electron energy on large number of non-bonding molecular orbitals, *J. Serb. Chem. Soc.* **69** (2004) 000-000.

[14] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, *Topics Curr. Chem.* **42** (1973) 49-93.

[15] I. Gutman, L. Türker, Angle of graph energy – A spectral measure of resemblance of isomeric molecules, *Indian J. Chem.* **42A** (2003) 2698-2701.

[16] I. Gutman, L. Türker, Correcting the azimutal angle concept: nonexistence of an upper bound, *Chem. Phys. Lett.* **378** (2003) 425-427.

[17] I. Gutman, L. Türker, Estimating the angle of total π-electron energy, *J. Mol. Struct. (Theochem)* **668** (2004) 119-121.

[18] F. Harary, *Graph Theory*, Addison-Wesley, Reading, 1969.

[19] Y. Hou, Unicyclic graphs with minimal energy, *J. Math. Chem.* **29** (2001) 163-168.

[20] Y. Hou, I. Gutman, C. W. Woo, Unicyclic graphs with maximal energy, *Lin. Algebra Appl.* **356** (2002) 27-36.

[21] J. Koolen, V. Moulton, Maximal energy graphs, *Adv. Appl. Math.* **26** (2001) 47-52.

[22] J. Koolen, V. Moulton, Maximal energy bipartite graphs, *Graph. Combin.* **19** (2003) 131-135.

[23] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Equienergetic graphs, *Kragujevac J. Math.* **26** (2004) 00-00.

[24] H. Sachs, Über selbstkomplementäre Graphen, *Publ. Math. Debrecen* **9** (1962) 270-288.

[25] H. Sachs, Über Teiler, Faktoren und charackteristische Polynome von Graph-en, Teil II, *Wiss. Z. Techn. Hochsch. Ilmeanau*, **13** (1967) 405-412.

[26] D. Stevanović, Energy and NEPS of graphs, *Lin. Multilin. Algebra*, in press.

[27] L. Türker, Mystery of the azimuthal angle of alternant hydrocarbons, *J. Mol. Struct. (Theochem)* **587** (2002) 123-127.

[28] L. Türker, On the mystery of the azimuthal angle of alternant hydrocarbons – an upper bound, *Chem. Phys. Lett.* **364** (2002) 463-468.

[29] H. B. Walikar, I. Gutman, P. R. Hampiholi, H. S. Ramane, Nonhyperenergetic graphs, *Graph Theory Notes New York* **51** (2001) 14-16.

[30] H. B. Walikar, H. S. Ramane, P. R. Hampiholi, On the energy of a graph, in: R. Balakrishnan, H. M. Mulder, A. Vijaykumar (Eds.), *Graph Connections*, Allied Publishers, New Delhi, 1999, pp. 120-123.

[31] H. B. Walikar, H. S. Ramane, P. R. Hampiholi, Energy of trees with edge independence number three, in: R. Nadarajan, P. R. Kandasamy (Eds.) *Mathema-tical and Computational Models*, Allied Publishers, New Delhi, 2001, pp. 306-312.

[32] B. Zhou, The energy of a graph, *MATCH Commun. Math. Comput. Chem.* **51** (2004) 111-118.

[33] B. Zhou, On the energy of a graph, *Kragujevac J. Sci.* **26** (2004) 5-12.