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ROTATING OSCILLATORY MHD POISEUILLE FLOW: AN EXACT SOLUTION

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ABSTRACT. An unsteady magnetohydrodynamics (MHD) Poiseuille flow in a rotating system is analyzed. The viscous, incompressible and electrically conducting fluid is flowing in between two infinite horizontal plates. The fluid is injected with constant velocity through the lower porous stationary plate and simultaneously removed through the upper porous plate oscillating in its own plane. The fluid in the channel is also acted upon by a pressure gradient varying periodically with time. The entire system rotates about an axis perpendicular to the planes of the plates. A magnetic field of uniform strength is applied along the axis of rotation. The induced magnetic field is neglected due to the assumption of a very small Reynolds number. An exact solution of the flow problem is obtained and the effects of various flow parameters on the velocity field and the skin-friction are illustrated through graphs. The effects of Coriolis force due to rotation of the system and the Lorentz force due to the applied magnetic field are discussed.

Keywords: Poiseuille, Porous, Magnetohydrodynamic (MHD), Rotating, Oscillatory.

INTRODUCTION

The introduction of electromagnetic to the fluid dynamics establishes new physical phenomena. The effects of transversely applied magnetic field on the flow of an electrically conducting viscous incompressible fluid have been discussed widely because of its astrophysical, geophysical and engineering applications. In astrophysics and geophysics it is applied to study the stellar and solar structures, interplanetary and inter stellar matter, solar storms and flares, radio propagation, through the ionosphere etc. In engineering it finds its application in MHD generators, ion propulsion, MHD bearings, MHD pumps, MHD boundary layer control of reentry vehicles etc. On account of their varied importance, these flows have been studied by several scholars. Notable amongst them are CRAMMER and PAI [1], FERRARO and PLUMPTON [2], SHERCLIFF [3]. Exact solutions to the classical problems of hydromagnetic channel flows for different simplified situations have been discussed by many scholars, e.g. ALPHER [4], NIGAM and SINGH [5] etc. Magnetohydrodynamic flow in a duct has been studied by CHANG and LUNDGREN [6]. YEN and CHANG [7] analyzed the effects of wall conductance on the magnetohydrodynamic Couette flow. ATTIA and KOTB [8] studied the MHD flow between two parallel porous plates.

In recent years a number of studies have appeared in the literature on the fluid phenomena on earth involving rotation to a greater or lesser extent. Several investigations have been carried out on various types of flows in a rotating system. JANA and DUTTA [9] studied "Couette flow" and heat transfer in a rotating system. VIDYANIDHU and NIGAM [10] analyzed the "Poiseuille flow" in a rotating system. The effect of uniform transverse magnetic field with or without suction was investigated by GUPTA [11]. Injection/suction effects for the case of rotating horizontal porous plates have also been studied by SOUNDALGEKAR [12] and GUPTA [13]. ATTIA [14] analyzed the effects of suction and injection on unsteady coquette flow with variable properties. The flows of fluids in rotating channels have been studied by a number of scholars viz. MAZUMDER [15], GANAPATHY [16], SINGH [17]. SINGH and MATHEW [18] also obtained exact periodic solution of an oscillatory MHD flow in a rotating horizontal porous channel.

The aim of the present paper is to obtain yet another exact solution of a rotating oscillatory MHD "Poiseuille flow in the presence of transverse magnetic field. The two infinite insulated porous walls of the channel are subjected to constant injection and suction. The upper plate is oscillating in its own plane and the pressure gradient drops periodically with time.

FORMULATION AND SOLUTION OF THE PROBLEM

Consider the flow of a viscous, incompressible and electrically conducting fluid in a rotating horizontal channel. In order to derive basic equations for the problem under consideration following assumptions are made:

(i) The flow considered is unsteady and laminar.
(ii) The fluid is finitely conducting and with constant physical properties.
(iii) A magnetic field of uniform strength is applied normal to the flow.
(iv) The magnetic Reynolds number is taken to be small enough so that the induced magnetic field is neglected.
(v) Hall effect, electrical and polarization effects are neglected.
(vi) The entire system (consisting of channel plates and the fluid) rotates about an axis perpendicular to the plates.

Under these assumptions we write hydromagnetic governing equations of motion and continuity in a rotating frame of reference as:

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\nabla\boldsymbol{V}) + 2\boldsymbol{\Omega}\times\boldsymbol{V} = -\frac{1}{\rho}\nabla p^{*} + \vartheta\nabla^{2}\boldsymbol{V} + \frac{1}{\rho}(\boldsymbol{J}\times\boldsymbol{B}), \tag{1}$$

$$\nabla.\boldsymbol{V} = 0. \tag{2}$$

In equation (1) the last term on the left hand side is the Coriolis force. On the right hand side of (1) the last term accounts for the Lorentz force due to magnetic field **B** and is given by

$$\boldsymbol{J}\times\boldsymbol{B} = \sigma(\boldsymbol{V}\times\boldsymbol{B})\times\boldsymbol{B}, \tag{3}$$

and the modified pressure $p^{*} = p' - \frac{\rho}{2}|\boldsymbol{\Omega}\times\boldsymbol{R}|^{2}$, where **R** denotes the position vector from the axis of rotation, p' denotes the fluid pressure, $\boldsymbol{J}$ is the current density and all other quantities have their usual meaning and have been defined in the text time to time.

Consider an unsteady flow of a viscous incompressible and electrically conducting fluid in a porous channel. The two infinite horizontal porous plates are distance '2d' apart. A coordinate system is chosen such that the X^{*} -axis is oriented along the centerline of the channel and Z^*-axis taken perpendicular to the planes of the plates lying in $z^* = \pm d$ planes as shown in Figure 1.

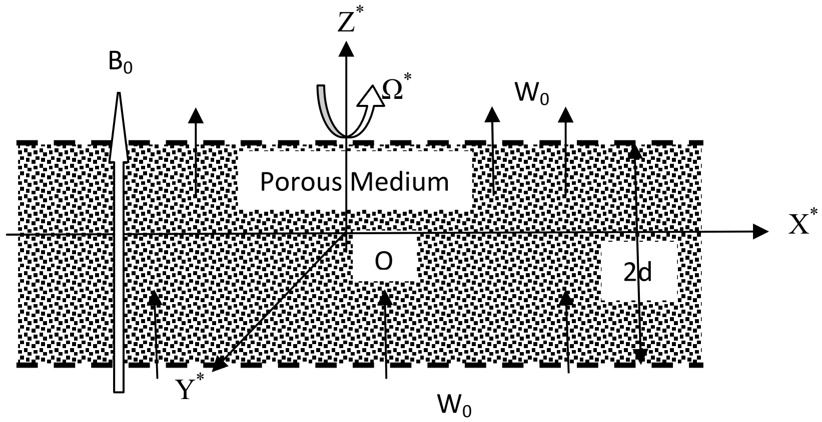

Fig.1. Geometrical presentation of the problem.

The lower stationary porous plate is subjected to a constant injection velocity 'w_0' while the upper porous plate oscillating in its own plane is subjected to the same constant but suction velocity 'w_0'. The Z^*- axis is considered to be the axis of rotation about which the fluid and the plates are assumed to be rotating as a solid body with a constant angular velocity Ω^*. A transverse magnetic field of uniform strength **B** $(0, 0, B_0)$ is also applied along the axis of rotation. The value of the magnetic Reynolds number is taken to be small enough so that the induced magnetic field is neglected. Hall effect, electrical and polarization effects are also neglected. Since the plates of the channel are infinite in X^* and Y^* directions and are electrically non-conducting, all physical quantities except the pressure depend on z^* and t^* only. The velocity may reasonably be assumed with its components along x^*, y^*, z^* directions as **V** (u^*, v^*, w^*). The magnetohydrodynamic (MHD) flow in the rotating porous channel is governed by the following equations:

$$\frac{\partial w^*}{\partial z^*} = 0 , \tag{4}$$

$$\frac{\partial u^*}{\partial t^*} + w^* \frac{\partial u^*}{\partial z^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \vartheta \frac{\partial^2 u^*}{\partial z^{*2}} + 2\Omega^* v^* - \frac{\sigma B_0^2}{\rho} u^* , \tag{5}$$

$$\frac{\partial v^*}{\partial t^*} + w^* \frac{\partial v^*}{\partial z^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y^*} + \vartheta \frac{\partial^2 v^*}{\partial z^{*2}} - 2\Omega^* u^* - \frac{\sigma B_0^2}{\rho} v^* , \tag{6}$$

$$0 = -\frac{1}{\rho}\frac{\partial p^*}{\partial z^*}, \tag{7}$$

where ρ is the density, ϑ is the kinematic viscosity, p^* is the modified pressure, t is the time. Equation (7) shows the constancy of the hydrodynamic pressure along the axis of rotation. We shall assume now that the fluid flows under the influence of pressure gradient varying periodically with time only in the X^*-axis is of the form

$$-\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} = P \cos\omega^* t^*, \tag{8}$$

where P is a constant.

The boundary conditions for the problem are

$$z^* = d;\ u^* = u^*(t^*) = U \cos\omega^* t^*, v^* = 0,\ w^* = w_0 , \tag{9}$$

$$z^* = -d;\ u^* = v^* = 0 \quad w^* = w_0, \tag{10}$$

where U is the mean velocity of the upper plate, '*' represents the dimensional physical quantities.
The integration of (4) under the boundary conditions (9) and (10) give

$$w^* = w_0 \ . \tag{11}$$

Substituting (11) and introducing the following non-dimensional quantities into equations (5) and (6)

$$\eta = \frac{z^*}{d}\ ,\ x = \frac{x^*}{d},\ y = \frac{y^*}{d}\ ,\ u = \frac{u^*}{U}\ ,\ v = \frac{v^*}{U},\ t = \omega^* t,^*\ \ p = \frac{p^*}{\rho U w_0}\ , \tag{12}$$

we get

$$\omega \frac{\partial u}{\partial t} + s \frac{\partial u}{\partial \eta} = -s \frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} + 2\Omega v - M^2 u\ , \tag{13}$$

$$\omega \frac{\partial v}{\partial t} + s \frac{\partial v}{\partial \eta} = \frac{\partial^2 v}{\partial \eta^2} - 2\Omega u - M^2 v\ \ , \tag{14}$$

where

$\omega = \frac{\omega^* d^2}{\vartheta}$ is the frequency of oscillations,
$\Omega = \frac{\Omega^* d^2}{\vartheta}$ is the rotation parameter,
$M = B_0 d \sqrt{\frac{\sigma}{\mu}}$ is the Hartmann number,
$s = \frac{w_0 d}{\vartheta}$ is the injection/suction parameter,
$\omega = \frac{\omega^* d^2}{\vartheta}$ is the frequency of oscillations.

The boundary conditions in the dimensionless form become

$$\eta = 1; \quad u = \cos t,\ v = 0\ , \tag{15}$$

$$\eta = -1; \quad u = v = 0\ . \tag{16}$$

For the oscillatory internal flow we shall assume that the fluid flows under the influence of a non-dimension pressure gradient oscillating in the direction of x-axis only which is of the form

$$-\frac{\partial p}{\partial x} = P\, cost. \tag{17}$$

By introducing a complex function F = u + iv we now combine equations (13) and (14) into single equation as

$$\omega \frac{\partial F}{\partial t} + s \frac{\partial F}{\partial \eta} = -s \frac{\partial p}{\partial x} + \frac{\partial^2 F}{\partial \eta^2} - (M^2 + 2i\Omega) F\ , \tag{18}$$

with boundary conditions

$$\eta = 1; \quad F = \cos t\ , \tag{19}$$

$$\eta = -1; \quad F = 0\ . \tag{20}$$

In order to solve equation (18) under boundary conditions (19) and (20) we assume the pressure gradient and $F(\eta, t)$ in complex notation as

$$F(\eta, t) = F_0(\eta) e^{it},\ -\frac{\partial p}{\partial x} = P e^{it}\ , \tag{21}$$

along with the boundary conditions in complex notation as

$$\eta = 1; \quad F = e^{it}\ , \tag{22}$$

$$\eta = -1; \quad F = 0\ . \tag{23}$$

Substituting expressions (21) in equation (18), we get

$$\frac{d^2F_0}{d\eta^2} - s\frac{dF_0}{d\eta} - l^2F_0 = -sP \ , \qquad (24)$$

with transformed boundary conditions as

$$\eta = 1; \quad F_0 = 1 \ , \qquad (25)$$

$$\eta = -1; \ F_0 = 0 \ . \qquad (26)$$

where $l^2 = (M^2 + 2i\Omega + i\omega)$.

The ordinary differential equation (24) is solved under the boundary conditions (25) and (26) to obtained the velocity field as

$$\mathrm{u}(\eta, t) = \left[\frac{sP}{l^2} + \frac{1}{\sinh(m-n)}\left\{\frac{e^{m\eta-n} - e^{n\eta-m}}{2} + \frac{sP}{l^2}\left(e^{m\eta}\sinh n - e^{n\eta}\sinh m\right)\right\}\right]e^{it} \ , \qquad (27)$$

where $m = \frac{s+\sqrt{s^2+4l^2}}{2}$, and $n = \frac{s-\sqrt{s^2+4l^2}}{2}$.

Knowing the velocity field the shear stress at the lower plate can be obtained as

$$\tau = \left(\frac{\partial F_0}{\partial \eta}\right)_{\eta=-1} e^{it} = \frac{e^{it}}{\sinh(m-n)}\left\{\frac{(m-n)e^{-s}}{2}\frac{sP}{l^2}\left(me^{-m}\sinh n - ne^{-n}\sinh m\right)\right\} . \qquad (28)$$

RESULTS AND DISCUSSION

An analysis of the flow of an electrically conducting, viscous incompressible fluid in an infinite horizontal porous channel is carried out. The fluid and the channel rotate in unison about an axis perpendicular to the planes of the plates and a magnetic field of uniform strength is also applied along this axis of rotation. The equations governing the flow are solved exactly for the velocity field. The numerical values of the velocity field are evaluated and illustrated in Figures 2-6 to assess the effects of injection/suction parameter (s), Hartmann number (M), rotation parameter Ω, the pressure gradient (P), the frequency of oscillations ω and the time t.

The effects of the rotation parameter Ω on the velocity profiles are shown in Fig.2. It is observed that the velocity goes on decreasing with the increasing rotation of the system. This means that increasing Coriolis force retards the forward flow due to the pressure gradient and the motion of the plate. The variations of the velocity profiles with the Hartmann number M are presented in Fig.3. It is clear from this figure that the velocity increases with the increase of magnetic field strength. Physically it means that the Lorentz force enhances the flow of fluid when the system is in rotation. The effect of injection/suction parameter (s) is illustrated in Fig.4. It is evident that increasing injection/suction parameter leads to the increase in the velocity. The increase of the velocity due to the increase of pressure gradient is also evident from Fig.5. It is expected physically also that the increasing favorable pressure gradient accelerates the flow field. Fig.6 exhibits that the velocity decreases due to the increasing frequency of oscillations ω. The time dependence of the velocity profiles are depicted in Fig.7. The figure shows that the velocity profiles have damped form of oscillations terminating at the lower plate.

The shear stress amplitude $|F|$ is presented against the frequency of oscillations ω in Fig.8. It is very clear from this figure that the amplitude goes on decreasing with increasing frequency ω.In order to study the effects of various parameters every curve is compared with the basic curve III. Comparison of curves III &V reveal that as the rotation Ω of the system increases $|F|$ decreases. Similarly by comparing curves III & IV a decrease in the amplitude $|F|$ is also noticed with the increase of Hartmann number M. The pairs of the curves III & II and III & I indicate that due to the increase of injection/suction parameter s and the pressure gradient P the shear stress increases significantly. The phase angle φ of the shear stress is shown in Fig.9. It is obvious from this figure that there is always a phase lag since all the values of φ are negative. Phase lag decrease with the increase of Hartmann number M (curves III &IV) and the injection/suction parameter s (curves III & II). There is no change in phase lag due to the increase of pressure gradient P since the curves (III & I) showing the increase in P coincides with each other. However, the comparison of curve III & V register an increase in phase lag φ due to the increase in rotation Ω of the system.

CONCLUSIONS

The shape of the velocity profiles in Figures 2-6 clearly depicts the flow field of fluid in the channel due to the variation in the periodic pressure gradient and the oscillations of the upper wall of the channel. In the lower half of the channel the influence of pressure gradient is predominant while in the upper half the flow is affected more by the oscillatory motion of the upper plate. The parabolic character of the velocity profiles shift towards the lower stationary plate due to the rotation of the system. The increasing Coriolis force retards the forward flow due to the pressure gradient and the motion of the plate. Also the Lorentz force enhances the flow of fluid when the system is in rotation. As expected physically also the increasing favorable pressure gradient accelerates the flow field. The velocity profiles have damped form of oscillations terminating at the lower plate. The amplitude of the skin friction decreases with increasing Coriolis and Lorentz forces but increases with increasing injection/suction at the plates and the pressure gradient. There always remains a phase lag.

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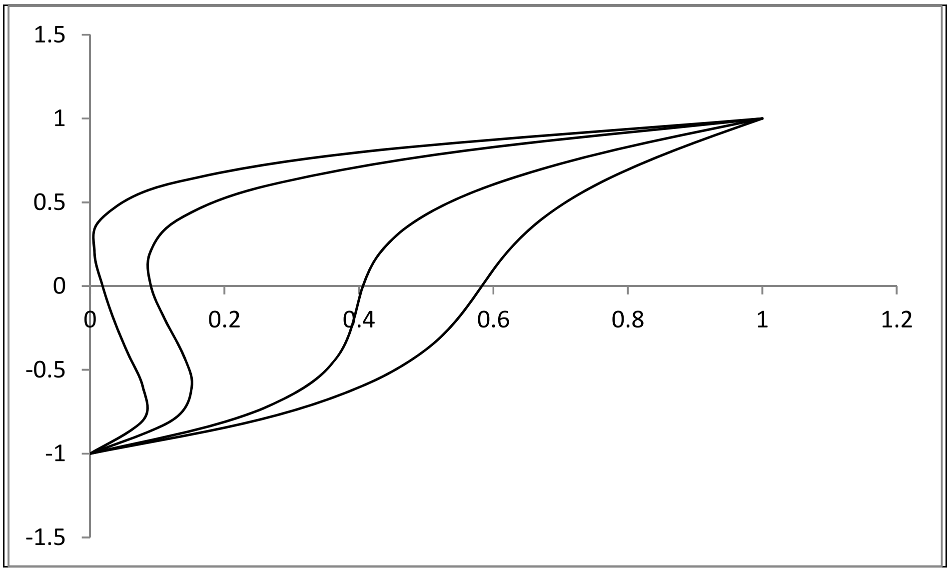

Fig. 2. Velocity profiles for M = 2, s = 1, P = 5, ω = 5 and t = 0.

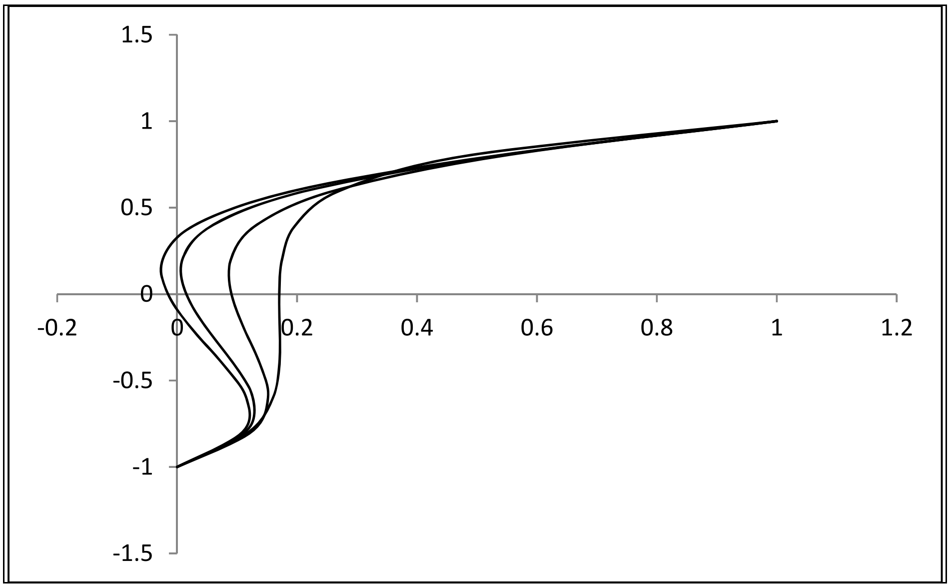

Fig. 3. Velocity profiles for Ω = 5, s = 1, P = 5, ω = 5 and t = 0.

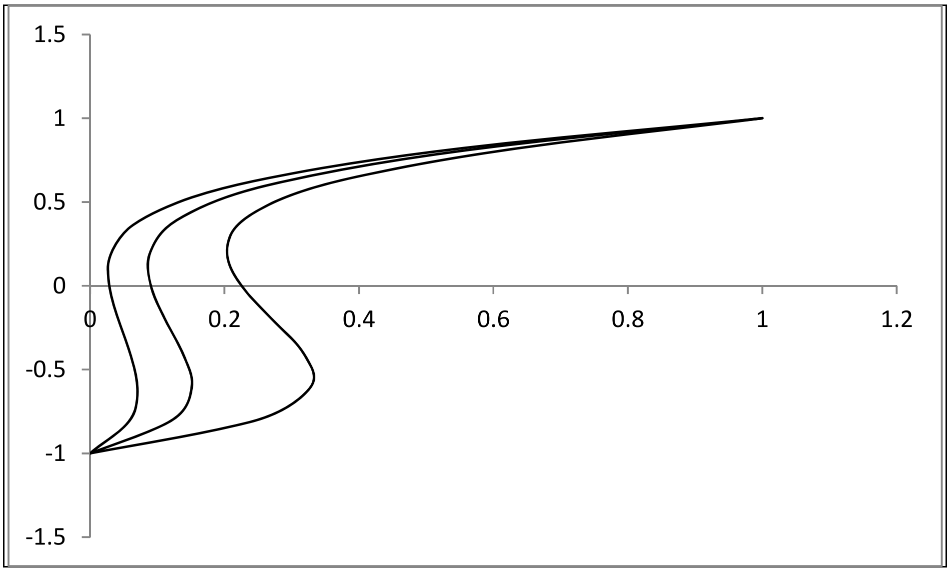

Fig. 4. Velocity profiles for $\Omega = 5$, M = 2, P = 5, $\omega = 5$ and t = 0.

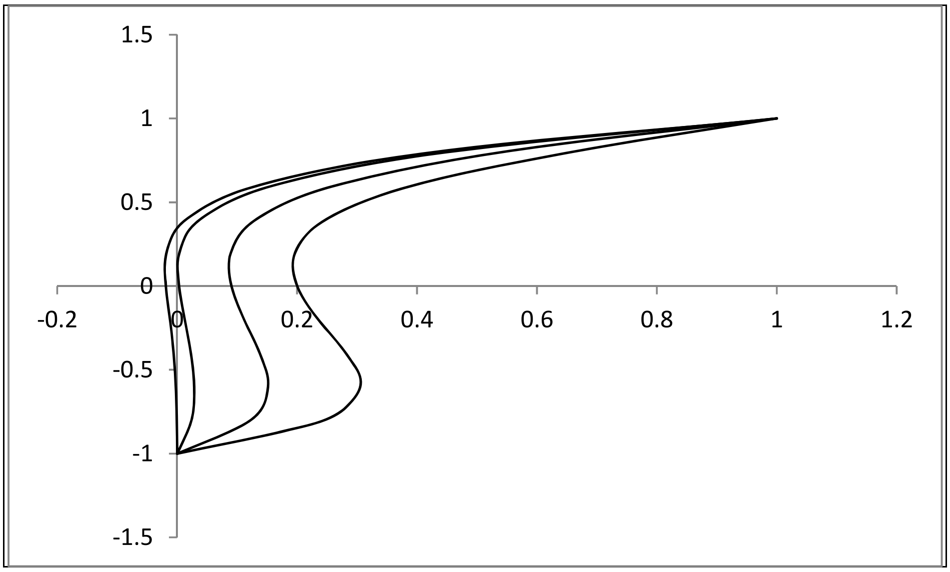

Fig. 5. Velocity profiles for $\Omega = 5$, M = 2, s= 1, $\omega = 5$ and t = 0.

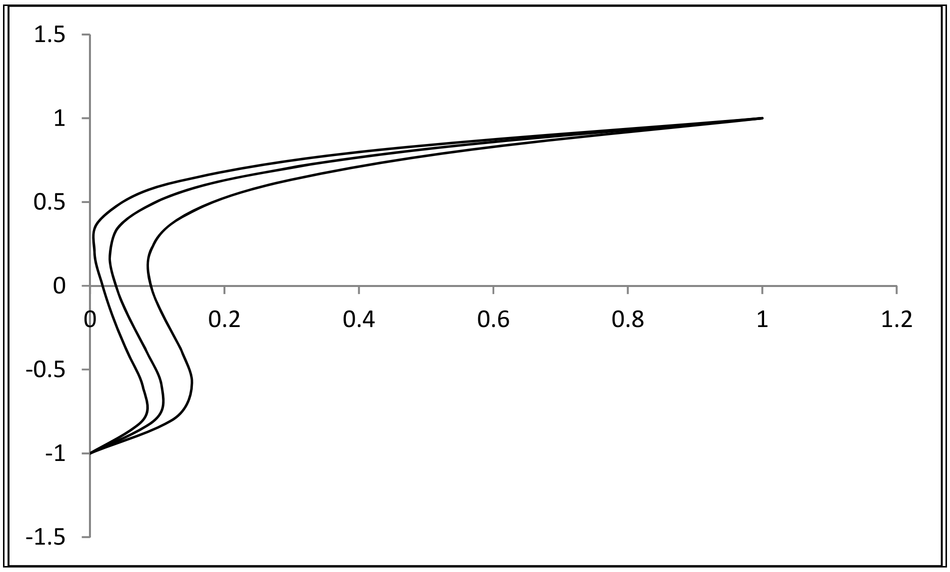

Fig. 6. Velocity profiles for Ω = 5, M = 2, s = 1, P = 5 and t = 0.

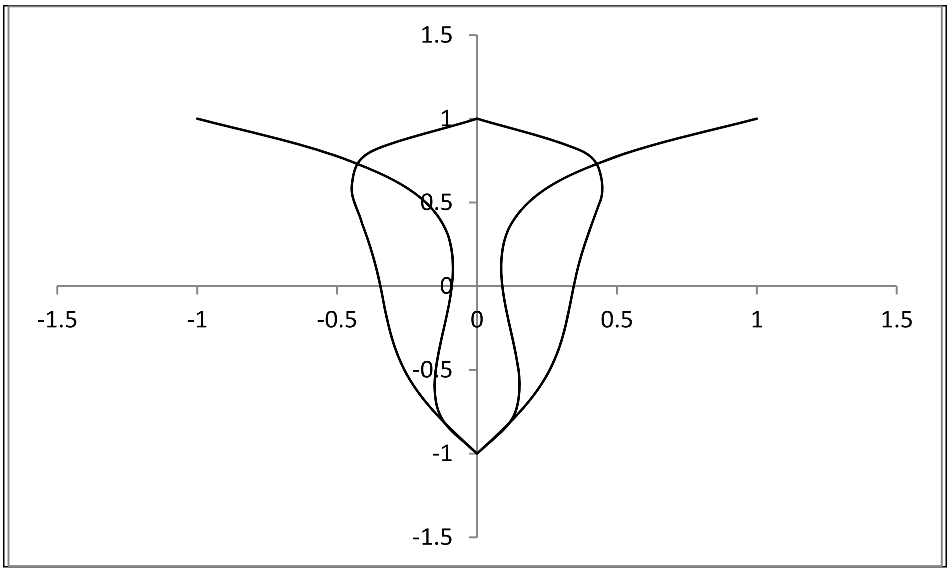

Fig. 7. Velocity profiles for Ω = 5, M = 2, s = 1, P = 5 and $\omega = 5$.

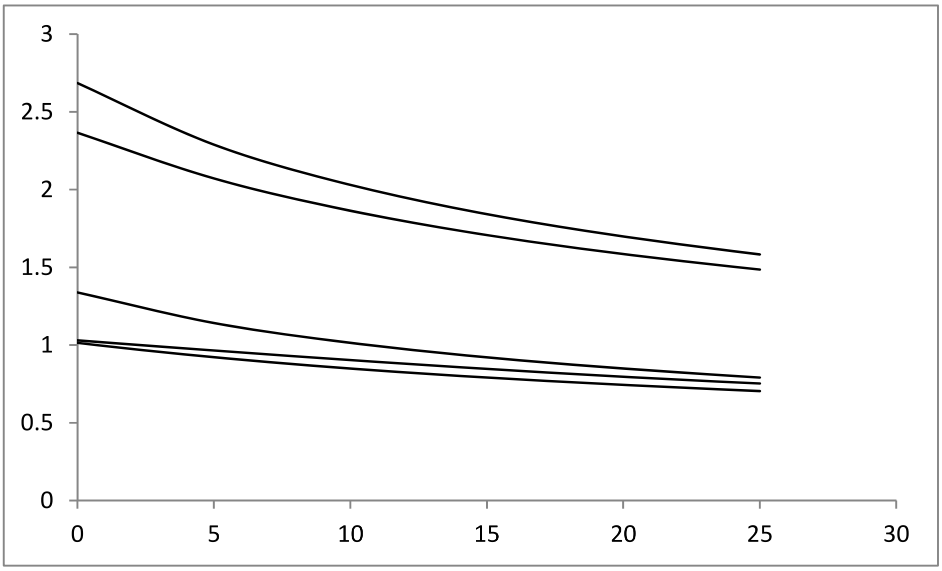

Fig. 8. The amplitude of the shear stress.

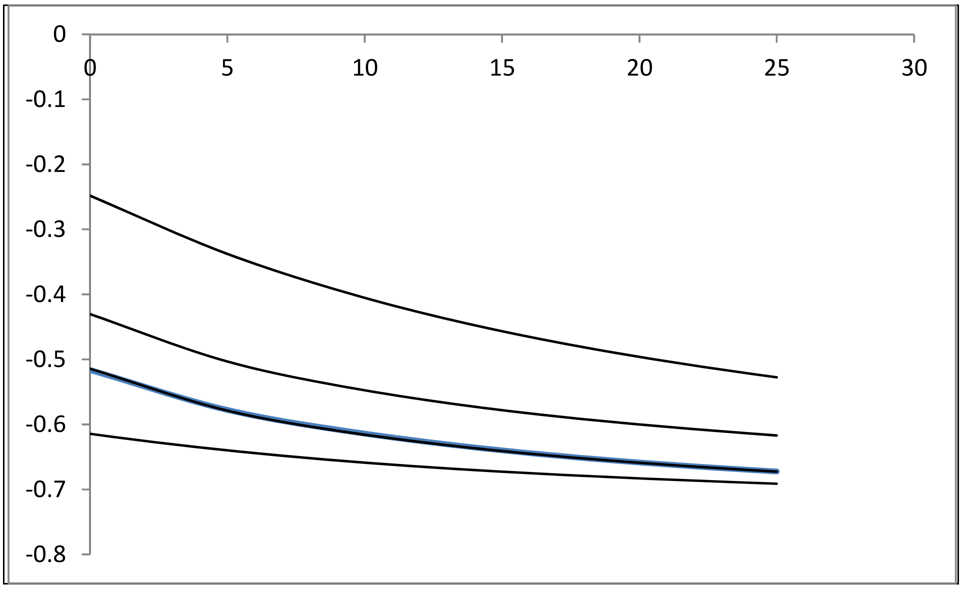

Fig. 9. The phase angle of the shear stress.