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COMPARISON OF DIRECT AND INDIRECT BOUNDARY ELEMENT METHODS FOR THE FLOW PAST A SPHERE

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ABSTRACT. In this paper, a comparison of direct and indirect boundary element methods is applied for calculating the flow past (i.e. velocity distribution) a sphere. To check the accuracy of the method, the computed flow velocity is compared with the analytical solution for the flow over the boundary of a sphere.

Keywords: boundary element methods, flow past, velocity distribution, sphere.

INTRODUCTION

From the time of fluid flow modeling, it had been struggled to find the solution of a complicated system of partial differential equations (PDE) for the fluid flows which needed more efficient numerical methods. With the passage of time, many numerical techniques such as finite difference method, finite element method, finite volume method and boundary element method etc. came into beings which made possible the calculation of practical flows. Due to discovery of new algorithms and faster computers, these methods were evolved in all areas in the past. These methods are CPU time and storage hungry. One of the advantages is that with boundary elements one has to discretize the entire surface of the body, whereas with domain methods it is essential to discretize the entire region of the flow field. The most important characteristics of boundary element method are the much smaller system of equations and considerable reduction in data which is prerequisite to run a computer program efficiently. Furthermore, this method is well–suited to problems with an infinite domain. From above discussion, it is concluded that boundary element method is a time saving, accurate and efficient numerical technique as compared to other numerical techniques which can be classified into direct boundary element method and indirect boundary element method. The direct method takes the form of a statement which provides the values of the unknown variables at any field point in terms of the complete set of all the boundary data. Whereas the indirect method utilizes a distribution of singularities over the boundary of the body and computes this distribution as the solution of integral equation.

FLOW PAST A SPHERE

Consider the flow past a sphere of radius a with center at the origin and let the onset flow be the uniform stream with velocity U in the positive direction of the z–axis, as shown in Fig. 1.

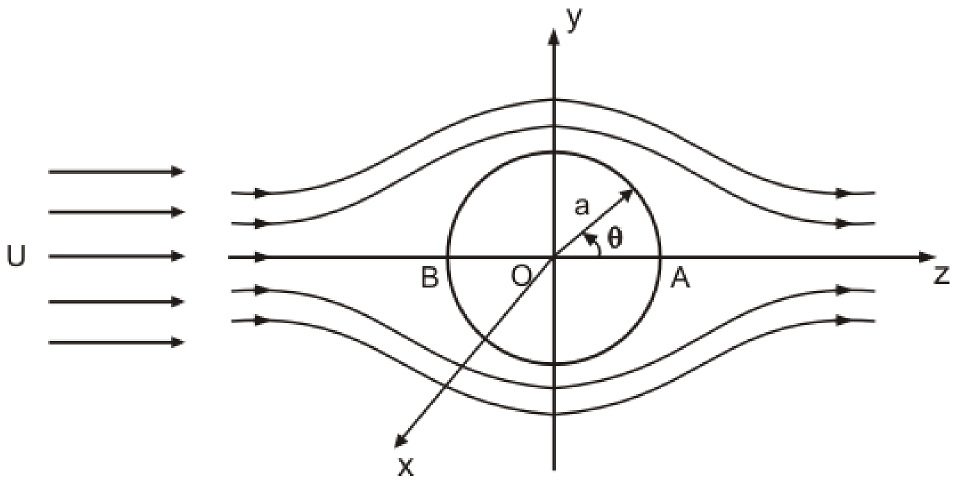


Fig. 1. – The flow past a sphere.

We know that the velocity potential and stream function due to the motion of a sphere of radius a moving with velocity U in the negative direction of z – axis in an infinite mass of liquid at rest at infinity are given by

$$\phi = -\frac{1}{2} U \frac{a^3}{r^2} \cos\theta , \quad \psi = \frac{1}{2} U \frac{a^3}{r} \sin^2\theta \tag{1}$$

If we now impress upon the sphere and the liquid , a uniform velocity field U in the positive direction of z – axis , the sphere will come to rest and the uniform stream which was at rest at infinity shall start moving with velocity U in the positive direction of z – axis . Thus we will get the streaming motion past a fixed sphere.

The superposition of the velocity field U in the positive direction of z – axis amounts to the addition of the term $-U r \cos\theta$ to the velocity potential and the term $-\frac{1}{2} U r^2 \sin^2\theta$ to the stream function in equations (1) . Thus the velocity potential for the uniform stream past a fixed sphere of radius a in the positive direction of z – axis becomes

$$\begin{aligned} \phi &= -U r \cos\theta - \frac{1}{2} U \frac{a^3}{r^2} \cos\theta \\ &= -U\left(r + \frac{a^3}{2 r^2} \right) \cos\theta \end{aligned} \tag{2}$$

The corresponding stream function for this flow can be written as

$$\begin{aligned} \psi &= -\frac{1}{2} U r^2 \sin^2\theta + \frac{1}{2} U \frac{a^3}{r} \sin^2\theta \\ &= -\frac{1}{2} U \left(r^2 - \frac{a^3}{r} \right) \sin^2\theta \end{aligned} \tag{3}$$

The velocity components at any point (r , θ) are given by

$$v_r = -\frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^3}{r^3} \right) \cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$$

The speed at any point in the flow field is given by

$$V = \sqrt{v_r^2 + v_\theta^2}$$

$$= U\sqrt{\left(1 - \frac{a^3}{r^3}\right)^2 \cos^2\theta + \left(1 + \frac{a^3}{2r^3}\right)^2 \sin^2\theta} \qquad (4)$$

The velocity components at any point (a, θ) on the surface of the sphere $r = a$ become

$$v_r = 0, \quad v_\theta = -\frac{3}{2}U\sin\theta \qquad (5)$$

Equation (5) shows that the velocity on the surface of the sphere is purely tangential . Therefore the speed at any point on the sphere itself is given by

$$V = \sqrt{v_r^2 + v_\theta^2}$$

$$= \sqrt{0 + \left(-\frac{3}{2}U\sin\theta\right)^2} = \frac{3}{2}U\sin\theta \qquad (6)$$

BOUNDARY CONDITIONS

The boundary condition to be satisfied over the surface of a sphere is

$$\frac{\partial \phi_{sphere}}{\partial n} = U(\hat{n} \cdot \hat{k}) \qquad (7)$$

where ϕ_{sphere} is the perturbation velocity potential of a sphere and $\hat{n}$ is the outward drawn unit normal to the surface of a sphere

Let $f(x, y, z) = x^2 + y^2 + z^2 - a^2$

Then $\nabla f = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$

Therefore $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}}{\sqrt{(2x)^2 + (2y)^2 + (2x)^2}}$

Thus $\hat{n} \cdot \hat{k} = \dfrac{2z}{\sqrt{(2x)^2 + (2y)^2 + (2x)^2}}$

$$= \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

The equation of the surface of the sphere is

$x^2 + y^2 + z^2 = 1$, where the radius a is taken as 1.

Therefore, the boundary condition in (7) takes the form

$$\frac{\partial \phi_{sphere}}{\partial n} = U z$$

$$= z, \quad (\text{Taking } U = 1) \qquad (8)$$

Equation (8) is the boundary condition which must be satisfied over the boundary of a sphere.

DISCRETIZATION OF ELEMENTS

Consider the surface of the sphere in one octant to be divided into three quadrilateral elements by joining the centroid of the surface with the mid points of the curves in the coordinate planes as shown in Fig. 2., then each element is divided further into four elements by joining the centroid of that element with the mid–point of each side of the element. Thus one octant of the surface of the sphere is divided into 12 elements and the whole surface of the body is divided into 96 boundary elements. The above mentioned method is adopted in order to produce a uniform distribution of element over the surface of the body.

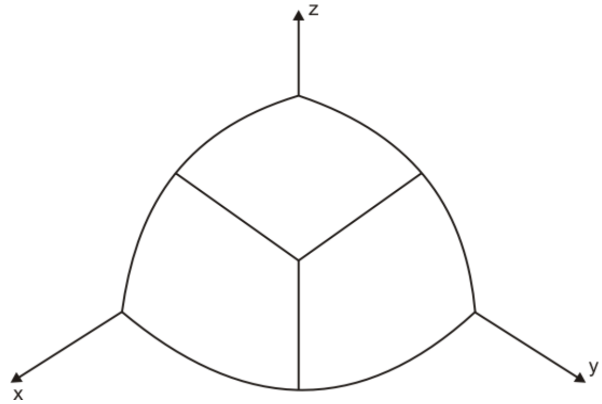


Fig. 2. – Surface of the sphere divided into three quadrilateral elements.

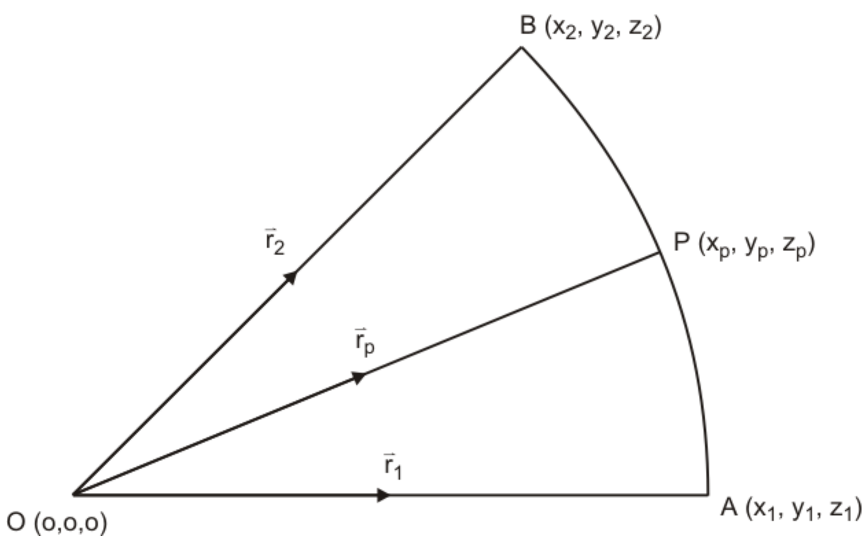


Fig. 3. - The method for finding the coordinate (x_p, y_p, z_p) of any point P on the surface of the sphere.

From Fig. 3 we have the following equation

$$|\vec{r}_p| = 1$$

$$\vec{r}_p \cdot \vec{r}_1 = \vec{r}_p \cdot \vec{r}_2$$

$$(\vec{r}_1 \times \vec{r}_2) \cdot \vec{r}_p = 0$$

or in cartesian form

$$x_p^2 + y_p^2 + z_p^2 = 1$$

$$x_p (x_1 - x_2) + y_p (y_1 - y_2) + z_p (z_1 - z_2) = 0$$

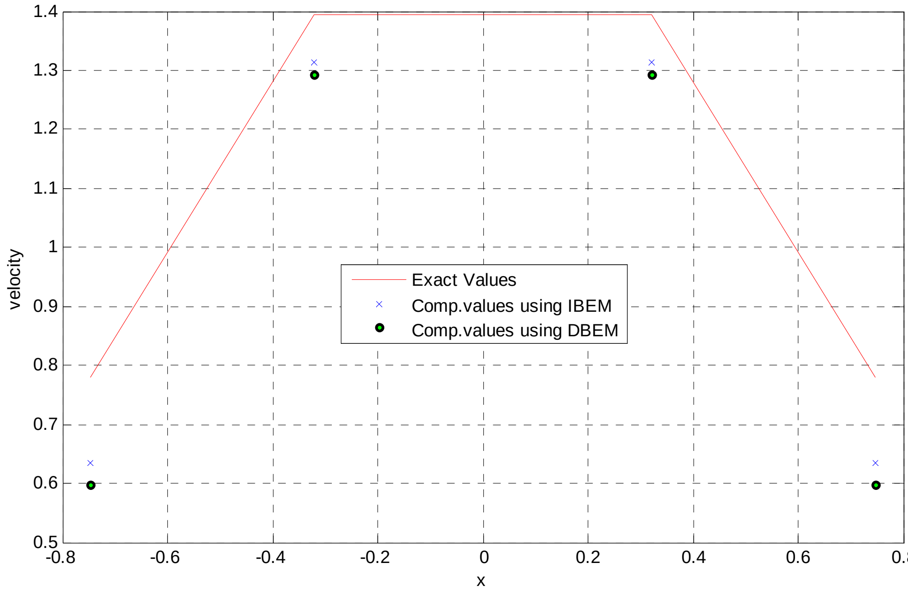
$$x_p (y_1 z_2 - z_1 y_2) + y_p (x_2 z_1 - x_1 z_2) + z_p (x_1 y_2 - x_2 y_1) = 0$$

As the body possesses planes of symmetry, this fact may be used in the input to the program and only the non–redundant portion need be specified by input points. The other portions are automatically taken into account. The planes of symmetry are taken to be the coordinate planes of the reference coordinate system. The advantage of the use of symmetry is that it reduces the order of the resulting system of equations and consequently reduces the computing time in running a program. As a sphere is symmetric with respect to all three coordinate planes of the reference coordinate system, only one eighth of the body surface need be specified by the input points, while the other seven–eighth can be accounted for by symmetry.

The calculated velocity distributions are compared with analytical solutions for the sphere using Fortran programming.

Table 1. – The comparison of the computed velocities with exact velocity over the surface of a sphere using 24 boundary elements.

ELEMENT	XM	YM	ZM	COMPUTED VELOCITY USING DBEM	COMPUTED VELOCITY USING IBEM	EXACT VELOCITY
1	-.321E+00	-.748E+00	.321E+00	.12920E+01	.13129E+01	.13953E+01
2	-.748E+00	-.321E+00	.321E+00	.59757E+00	.63485E+00	.77853E+00
3	-.748E+00	.321E+00	.321E+00	.59757E+00	.63485E+00	.77853E+00
4	-.321E+00	.748E+00	.321E+00	.12920E+01	.13129E+01	.13953E+01
5	.321E+00	.748E+00	.321E+00	.12920E+01	.13129E+01	.13953E+01
6	.748E+00	.321E+00	.321E+00	.59757E+00	.63485E+00	.77853E+00
7	.748E+00	-.321E+00	.321E+00	.59757E+00	.63485E+00	.77853E+00
8	.321E+00	-.748E+00	.321E+00	.12920E+01	.13129E+01	.13953E+01
9	-.321E+00	-.321E+00	.748E+00	.12920E+01	.13129E+01	.13953E+01
10	-.321E+00	.321E+00	.748E+00	.12920E+01	.13129E+01	.13953E+01
11	.321E+00	.321E+00	.748E+00	.12920E+01	.13129E+01	.13953E+01
12	.321E+00	-.321E+00	.748E+00	.12920E+01	.13129E+01	.13953E+01

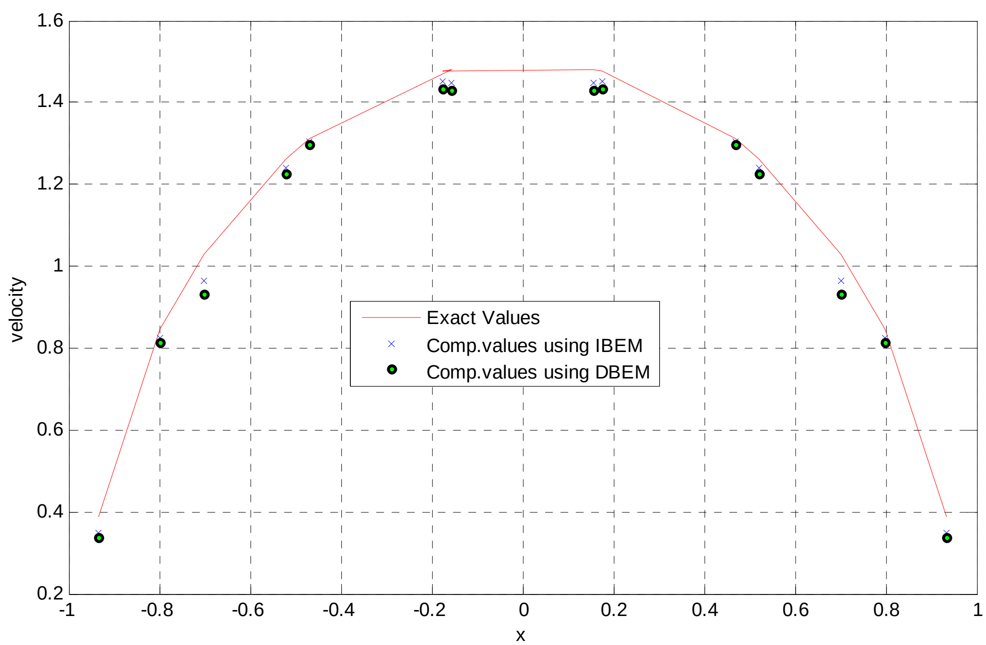


Graph 1. - Comparison of computed and analytical velocity distributions over the surface of a sphere using 24 boundary elements.

Table 2. – The comparison of the computed velocities with exact velocity over the surface of a sphere using 24 boundary elements.

ELEMENT	XM	YM	ZM	COMPUTED VELOCITY USING DBEM	COMPUTED VELOCITY USING IBEM	EXACT VELOCITY
1	-.177E+00	-.934E+00	.177E+00	.14326E+01	.14526E+01	.14747E+01
2	-.522E+00	-.798E+00	.157E+00	.12258E+01	.12418E+01	.12623E+01
3	-.798E+00	-.522E+00	.157E+00	.81448E+00	.82398E+00	.84609E+00
4	-.934E+00	-.177E+00	.177E+00	.33929E+00	.34768E+00	.38819E+00
5	-.934E+00	.177E+00	.177E+00	.33929E+00	.34768E+00	.38819E+00
6	-.798E+00	.522E+00	.157E+00	.81448E+00	.82398E+00	.84609E+00
7	-.522E+00	.798E+00	.157E+00	.12258E+01	.12418E+01	.12623E+01
8	-.177E+00	.934E+00	.177E+00	.14326E+01	.14526E+01	.14747E+01
9	.177E+00	.934E+00	.177E+00	.14326E+01	.14526E+01	.14747E+01
10	.522E+00	.798E+00	.157E+00	.12258E+01	.12418E+01	.12623E+01
11	.798E+00	.522E+00	.157E+00	.81448E+00	.82398E+00	.84609E+00
12	.934E+00	.177E+00	.177E+00	.33929E+00	.34768E+00	.38819E+00
13	.934E+00	-.177E+00	.177E+00	.33929E+00	.34768E+00	.38819E+00
14	.798E+00	-.522E+00	.157E+00	.81448E+00	.82398E+00	.84609E+00
15	.522E+00	-.798E+00	.157E+00	.12258E+01	.12418E+01	.12623E+01
16	.177E+00	-.934E+00	.177E+00	.14326E+01	.14526E+01	.14747E+01
17	-.157E+00	-.798E+00	.522E+00	.14314E+01	.14495E+01	.14801E+01
18	-.470E+00	-.703E+00	.470E+00	.12987E+01	.13038E+01	.13113E+01
19	-.703E+00	-.470E+00	.470E+00	.93296E+00	.96588E+00	.10301E+01
20	-.798E+00	-.157E+00	.522E+00	.81448E+00	.82398E+00	.84609E+00
21	-.798E+00	.157E+00	.522E+00	.81448E+00	.82398E+00	.84609E+00
22	-.703E+00	.470E+00	.470E+00	.93296E+00	.96588E+00	.10301E+01
23	-.470E+00	.703E+00	.470E+00	.12987E+01	.13038E+01	.13113E+01
24	-.157E+00	.798E+00	.522E+00	.14314E+01	.14495E+01	.14801E+01

25	.157E+00	.798E+00	.522E+00	.14314E+01	.14495E+01	.14801E+01
26	.470E+00	.703E+00	.470E+00	.12987E+01	.13038E+01	.13113E+01
27	.703E+00	.470E+00	.470E+00	.93296E+00	.96588E+00	.10301E+01
28	.798E+00	.157E+00	.522E+00	.81448E+00	.82398E+00	.84609E+00
29	.798E+00	-.157E+00	.522E+00	.81448E+00	.82398E+00	.84609E+00
30	.703E+00	-.470E+00	.470E+00	.93296E+00	.96588E+00	.10301E+01
31	.470E+00	-.703E+00	.470E+00	.12987E+01	.13038E+01	.13113E+01
32	.157E+00	-.798E+00	.522E+00	.14314E+01	.14495E+01	.14801E+01
33	-.157E+00	-.522E+00	.798E+00	.14314E+01	.14495E+01	.14801E+01
34	-.470E+00	-.470E+00	.703E+00	.12987E+01	.13038E+01	.13113E+01
35	-.522E+00	-.157E+00	.798E+00	.12258E+01	.12418E+01	.12623E+01
36	-.522E+00	.157E+00	.798E+00	.12258E+01	.12418E+01	.12623E+01
37	-.470E+00	.470E+00	.703E+00	.12987E+01	.13038E+01	.13113E+01
38	-.157E+00	.522E+00	.798E+00	.14314E+01	.14495E+01	.14801E+01
39	.157E+00	.522E+00	.798E+00	.14314E+01	.14495E+01	.14801E+01
40	.470E+00	.470E+00	.703E+00	.12987E+01	.13038E+01	.13113E+01
41	.522E+00	.157E+00	.798E+00	.12258E+01	.12418E+01	.12623E+01
42	.522E+00	-.157E+00	.798E+00	.12258E+01	.12418E+01	.12623E+01
43	.470E+00	-.470E+00	.703E+00	.12987E+01	.13038E+01	.13113E+01
44	.157E+00	-.522E+00	.798E+00	.14314E+01	.14495E+01	.14801E+01
45	-.177E+00	-.177E+00	.934E+00	.14326E+01	.14526E+01	.14747E+01
46	-.177E+00	.177E+00	.934E+00	.14326E+01	.14526E+01	.14747E+01
47	.177E+00	.177E+00	.934E+00	.14326E+01	.14526E+01	.14747E+01
48	.177E+00	-.177E+00	.934E+00	.14326E+01	.14526E+01	.14747E+01



Graph 2. - Comparison of computed and analytical velocity distributions over the surface of a sphere using 96 boundary elements.

CONCLUSION

A direct and indirect boundary element methods have been applied for the calculation of flow past a sphere. The calculated flow velocities obtained using these methods are compared with the analytical solutions for flow over the boundary of a sphere. It is found that the results obtained with the indirect boundary element method for the flow past are excellent in agreement with the analytical results for the body under consideration.

Acknowledgement:

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