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# ASYMPTOTIC SOLUTION FOR HYDROMAGNETIC ROTATING DISK FLOW

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ABSTRACT. In this paper, we present asymptotic solution to a Navier-Stokes equation of von Karman type for the hydromagnetic flow of a conducting fluid due to a rotating disk. Asymptotic solutions to a Navier-Stokes equation is given in the case of small as well as large values of the magnetic interaction number $\beta$ whose coefficients are obtained in closed form in terms of properly scaled von Karman's similarity coordinate. Straining of coordinates is used to remove secular terms and enable to obtain expressions that can be used to determine the coefficients of the expansions to any order. A comparison of the asymptotic solution with an exact numerical solution for the governing nonlinear differential equations is presented.

## INTRODUCTION

The flow due to an infinite rotating disk is one of the classical problems in fluid mechanics which was first introduced by VON KARMAN (1921). The flow is a fully three dimensional one, involving a primary (azimuthal) flow and a secondary (meridional) flow. von Karman formulated the problem in the steady state and used similarity transformations to reduce the governing partial differential equations to ordinary differential equations. Asymptotic solutions were obtained for the reduced system of ordinary differential equations (COCHRAN, 1934). Their analysis was much simpler and valuable information was gained from it. This gave the problem significant theoretical value and invited many researchers to add to it new features. The extension of the steady hydrodynamic problem to the transient state was done by BENTON (1966). The effect of uniform suction or injection through a rotating porous disk on the steady hydrodynamic or hydromagnetic flow induced by the disk was investigated (STUART, 1954; KUIKEN, 1971; OCKENDON, 1972; ATTIA, 1998, 2001, 2002).

In this paper, asymptotic solutions of von Karman rotating disk problem of a steady flow of a viscous incompressible conducting fluid were obtained. The magnetic effect is to restrain the motion in the azimuthal and radial directions by imposing resisting force components that are proportional to the corresponding velocity components and the magnetic interaction number $\beta$ (SUTTON *et al*., 1965).

In particular, we study this problem in the limit as $\beta$ tends to zero or infinity. Straining the coordinate $\zeta$ is used to remove this difficulty (NAYFEH, 1973). Expressions that can be used to determine the coefficients of the expansions to any order are obtained. Using finite differences and linearization, an exact numerical solution for the governing nonlinear differential equations is represented which takes into account the asymptotic boundary conditions. The results of the asymptotic solution are compared with that of the exact numerical solution to check its range of validity. Sample results are presented.

## 2. The Governing Equations

We study the steady laminar flow of an incompressible viscous conducting fluid of density *ρ*, electrical conductivity $\sigma$, and kinematic viscosity *υ*. The motion is due to the rotation of an insulated disk of infinite extent about an axis perpendicular to its plane with constant angular speed *ω*. Otherwise the fluid is at rest under pressure $p_\infty$. A schematic diagram for the problem is shown in Fig. 1. A external uniform magnetic field of flux density $B_o$ directed parallel to the axis of rotation is applied. The equations of steady motion are given by

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\rho(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r})+\frac{\partial p}{\partial r}=\mu(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2})-\frac{\sigma B_o^2}{\omega}u \tag{2}$$

$$\rho(u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r})=\mu(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2})-\frac{\sigma B_o^2}{\omega}v \tag{3}$$

$$\rho(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z})+\frac{\partial p}{\partial z}=\mu(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}) \tag{4}$$

where *u*, *v*, *w* are velocity components in the directions of increasing $r, \varphi$, *z* respectively, *p* is denoting the pressure, $\mu$ is the coefficient of viscosity, $\rho$ is the density of the fluid. The last term in Eqs. (2) and (3) represents the components of the electromagnetic force (SUTTON *et al*., 1965).

The rotational symmetry suggests the use of cylindrical polar coordinates so that the flow variables (the radial, azimuthal, and normal velocity component *u*, *v*, and *w*, and the pressure *p*) be dependent only on the radial distance *r* and the normal distance *z*. Following VON KARMAN (1921) who discovered the self-similar nature of the problem, we introduce the similarity variables

$$\zeta=\frac{z}{\sqrt{\nu/\omega}}, \tag{5a}$$

measuring distances in the normal direction,

$$F(\zeta) = \frac{u}{r\omega}, G(\zeta) = \frac{v}{r\omega}, H(\zeta) = \frac{w}{\sqrt{r\omega}}, \quad \text{(5b,c,d)}$$

representing the radial, azimuthal, and normal velocity components, and

$$Q(\zeta) = \frac{(p - p_\infty)}{\rho\omega\nu}, \quad \text{(5e)}$$

representing the pressure.

Using von Karman transformations given by Eq. (5), the governing continuity and Navier-Stokes equations (1)-(4) reduce to the following set of ordinary differential equations

$$H' + 2F = 0, \quad \text{(6a)}$$

$$F'' - HF' - F^2 + G^2 - \beta F = 0, \quad \text{(6b)}$$

$$G'' - HG' - 2FG - \beta G = 0, \quad \text{(6c)}$$

$$H'' - HH' - Q' - \beta H = 0, \quad \text{(6d)}$$

where the primes denote differentiation with respect to $\zeta$, and $\beta = \sigma B_o^2 / \rho\omega$ is the magnetic interaction number (Sutton et al., 1965). The terms including $\beta$ represent the magnetic force. Equations (6) are supplemented with the no-injection and no-slip conditions

$$H(0) = 0, F(0) = 0, G(0) = 1 \quad \text{(6e,f,g)}$$

and the far-field conditions

$$\zeta \to \infty : F \to 0, G \to 0, Q \to 0 \quad \text{(6h,i,j)}$$

## 3. The Numerical Solution

The system of nonlinear ordinary differential equations is solved by a two-point finite difference technique. We write this system as a set of first order equations by introducing new independent variables and the resulting nonlinear equations are solved iteratively. The linearized equations have to be solved in the infinite domain $0 < \zeta < \infty$. A finite domain $0 < \zeta < \zeta_f$ is used instead with $\zeta_f$ chosen large enough to ensure that the solutions are not affected by imposing the asymptotic conditions at a finite distance. The computational domain is divided into (I) intervals where (I) is the number of $\zeta$ divisions. Finite difference equations relating these variables are obtained by writing the linearized equations at the midpoint of the computational cell and then replacing the different terms by their second order accurate central difference approximations. Extrapolation to zero step size has been performed to produce fourth order accurate solutions.

**4. Asymptotic solution for small *β***

Making use of Cochran's analysis (1934) of von Karman problem, we recast problem (6) in terms of Cochran's variables

$$\eta = \zeta / \sqrt{c}, h(\eta) = 1 + H\sqrt{c}, \quad (7a,b)$$

$$f(\eta) = Fc, g(\eta) = Gc, q(\eta) = Qc, \quad (7c,d,e)$$

where Cochran's parameter c is such that

$$\sqrt{c} = -1 / H(\infty) \quad (7f)$$

Note that $\sqrt{c}$ is real since $H(\infty)$ is negative representing an inflow toward the disk to compensate for the fluid that is expelled radially by the centrifugal effect.

The governing equations become

$$h' + 2f = 0, \quad (8a)$$

$$f'' - (h-1)f' - f^2 + g^2 - c\beta f = 0, \quad (8b)$$

$$g'' - (h-1)g' - 2fg - c\beta g = 0, \quad (8c)$$

$$h'' - (h-1)h' - q' - \beta c(h-1) = 0, \quad (8d)$$

where, now the primes denote differentiation with respect to $\eta$.

The boundary conditions take the form

$$h(0) = 1, f(0) = 0, g(0) = c, \quad (8e,f,g)$$

$$\eta \to \infty : h \to 0, f \to 0, g \to 0, q \to 0, \quad (8h,i,j,k)$$

where (8h) is implied by (7b,f). After Eqs. (8a)-(8c) have been solved, pressure $q$ from (8d) can be obtained by using the condition on $q$ from (8e,f,g)-(8h,i,j,k) and it is given by

$$q = h' - 2^{-1}h^2 + h - \beta c \int_0^{\eta} (h-1) d\eta$$

The terms including $\beta$ in Eqs. (8) are those expressing the magnetic effect. By setting $\beta$=0 we arrive at von Karman's problem (written in Cochran's variables). For small $\beta$ the terms $c\beta f$ and $c\beta g$ cause a perturbation $O(\beta)$ to von Karman's problem. The asymptotic solution to the new problem (8) would, therefore, be expected to have, in the leading order, von Karman's solution and to proceed in powers of $\beta$.

We introduce straightforward expansions of the form (summations without upper bounds run indefinitely)

$$(\ ) = \sum_{i=0} (\ )_i \beta^i \quad (9a)$$

for $h, f, g$, and $q$, while the expansion for $c$ takes the form

$$\{\ \} = \sum_{i=0} \{\ \}_i \beta^i \tag{9b}$$

Note that while $(\ )_i$'s depend on $\eta$, the $\{\ \}_i$'s are constants. Substituting in Eqs. (8) and equating the coefficients of like powers of $\beta$ result in a hierarchy of problems, the zeroth order one of which is von Karman's problem:

$$h_o' + 2f_o = 0, \tag{10a}$$

$$f_o'' - (h_o - 1)f_o' - f_o^2 + g_o^2 = 0, \tag{10b}$$

$$g_o'' - (h_o - 1)g_o' - 2f_o g_o = 0, \tag{10c}$$

$$h_o(0) = 1, f_o(0) = 0, g_o(0) = c_o, \tag{10d,e,f}$$

$$\eta \to \infty : h_o \to 0, f_o \to 0, g_o \to 0, \tag{10g,h,i,j}$$

Cochran's asymptotic expansions as $\eta \to \infty$ takes the form

$$(\ )_o = \sum_{j=1} (\ )_{o,j} e^{-j\eta} \tag{11}$$

Obviously, these expansions satisfy the farfield conditions (6h,i,j,k). Substituting in Eqs. (10a,b,c) and equating the coefficients of $e^{-j\eta}$, recurrence relations relating the $(\ )_{o,j}$'s in terms of two unknown coefficients $f_{o,1}$ and $g_{o,1}$. Cochran determined these two coefficients by patching the asymptotic solution (11) to the asymptotic solution as $\eta \to 0$ at an intermediate-point, and thus obtained a solution of problem (10). Benton (1966), on the other hand, determined the two coefficients by forcing the asymptotic solution (11) to satisfy (10d,e) and thus avoided the problems of patching and obtained an improved solution.

When Benton's method is extended to the higher order problems obtained by introducing expansions (9) into Eqs. (8), it is found that expansions of the form

$$(\ )_i = \sum_{j=1} (\ )_{i,j} e^{-j\eta} \tag{12a}$$

are not adequate for representing the solution of the $i$-th problem (when $i \geq 1$). Rather, expansions of the form

$$(\ )_i = \sum_{k=0}^{i} \left( \sum_{j=1} (\ )_{i,j,k} e^{-j\eta} \right) \eta^k \tag{12b}$$

for $h, f, g$, and $q$, with expansion (9b) for $c$, are to be used.

However, the terms including $e^{-j\eta}\eta^k$ for $k > 0$ are secular. They render the expansions non-uniformly valid as $\eta \to \infty$. To overcome this difficulty, straining of the coordinate $\eta$ is used. A strained coordinate $n$ defined by

$$n = s\eta \tag{13}$$

is introduced and the straining function is chosen such that no secular terms appear. The governing Eqs. (8) become

$$sh' + 2f = 0, \tag{14a}$$

$$s^2 f'' - s(h-1)f' - f^2 + g^2 - c\beta f = 0, \tag{14b}$$

$$s^2 g'' - s(h-1)g' - 2fg - c\beta g = 0, \tag{14c}$$

where the primes denote differentiation with respect to $n$, while the boundary conditions (12e-k) remain unchanged.

Transformation (13) is a near identity transformation with the straining function $s$ having unity as a leading term. This leads again to von Karman's problem as a leading order approximation to the problem defined by Eqs. (14). Its solution using Benton's method has been described above. One can now proceed to solve the next problem $O(\beta)$ using Benton's method determining, at the same time, the $O(\beta)$-term in $s$ that removes the secular behaviour and the process can, theoretically, be repeated with higher order problems. However, the analysis becomes more and more un-widely. Fortunately, the first few terms in the expansions show a systematic development that is exploited here to make possible the determination of any number of terms in the expansions. The expansions for $h$, $f$, $g$, and $q$ assume the form

$$(\ )_i = \sum_{i=0} \left( \sum_{j=1} (\ )_{i,j} e^{-jn} \right) \beta^i \tag{15}$$

while $c$ and $s$ have expansions of the form (9b).

The usual procedure of substituting and equating the coefficients of $e^{-jn}\beta^i$ is carried out. It leads to the following relations between the coefficients of the expansions, that guarantee satisfaction of the governing Eqs. (14) and the conditions as $n \to \infty$.

$$s_o = 1, s_1 = c_o \tag{16a,b}$$

$$s_i = c_{i-1} - \sum_{l=1}^{i-1} s_l s_{i-l}, i \geq 2 \tag{16c}$$

For $j \geq 2$ and $i \geq 1$

$$f_{o,j} = \frac{1}{j^2 - j} \sum_{m=1}^{j-1} \left( f_{o,m} f_{o,j-m} - g_{o,m} g_{o,j-m} - m f_{o,m} h_{o,j-m} \right) \tag{17a}$$

$$f_{i,j}=\frac{1}{j^2-j}\left(-\sum_{l=1}^{i}\left((j^2-j)s_l+(j^2-1)c_{l-1}\right)f_{i-l,j}+\sum_{l=0}^{i}\sum_{m=1}^{j-1}\left(f_{l,m}f_{i-l,j-m}-g_{l,m}g_{i-l,j-m}-mh_{i-l,j-m}\sum_{u=0}^{l}s_u f_{l-u,m}\right)\right)$$

(17b)

$$g_{o,j}=\frac{1}{j^2-j}\sum_{m=1}^{j-1}\left(2f_{o,m}g_{o,j-m}-mg_{o,m}h_{o,j-m}\right) \tag{18a}$$

$$g_{i,j}=\frac{1}{j^2-j}\left(\sum_{l=0}^{i}\sum_{m=1}^{j-1}\left(2f_{l,m}g_{i-l,j-m}-mh_{i-l,j-m}\sum_{u=0}^{l}s_u g_{l-u,m}\right)-\sum_{l=1}^{i}\left((j^2-j)s_l+(j^2-1)c_{l-1}\right)g_{i-l,j}\right) \tag{18b}$$

For $j\geq 1$

$$h_{o,j}=\frac{2f_{o,j}}{j} \tag{19a}$$

$$h_{i,j}=\frac{2f_{i,j}}{j}-\sum_{l=1}^{i}s_l h_{i-l,j}, i\geq 1 \tag{19b}$$

$$c_o=\sum_{j=1}g_{o,j} \tag{20a}$$

For $i\geq 0$, $c_i=\sum_{j=1}g_{i,j}, i\geq 1$ (20b)

The satisfaction of the conditions at $n$=0 gives the following relations.

$$\sum_{j=1}h_{i,j}=1, i=0 \tag{21a}$$

$$\sum_{j=1}h_{i,j}=0, i\geq 1 \tag{21b}$$

$$\sum_{j=1}f_{i,j}=0, i\geq 0 \tag{21c}$$

For every $i$, starting with $i$=0, we perform the following calculations, in the given order, noting that each calculation uses values obtained in previous ones.
(a) Calculate $s_i$ using Eqs. (16).
(b) Perform the following iterative scheme to calculate $h_{i,j}$, $f_{i,j}$ and $g_{i,j}$ for $j$=1 to $j$=$J$ where $J$ is a cutoff limit on the number of $j$-terms that is chosen high enough in order not to affect the accuracy.
1. Guess starting values for $f_{i,1}$ and $g_{i,1}$ then calculate the starting value for $h_{i,1}$ using Eqs. (19).

2. For every $j$, starting with $j$=2, calculate $f_{i,j}$ using Eqs. (13) and $g_{i,j}$ using Eqs. (18), then $h_{i,j}$ using Eqs. (19).

3. Calculate new values for $h_{i,1}$, $f_{i,1}$, and $g_{i,1}$ so as to satisfy Eqs. (18), using the following relations that are obtained by manipulating Eqs. (13), (15), and (18). Note that the values of $h_{i,1}$, $f_{i,1}$, and $g_{i,1}$ appearing in the right-hand sides are old values.

$$h_{o,1} = 1 - \sum_{j=2}^{J} h_{o,j} \tag{22a}$$

$$f_{o,1} = 0.5 h_{o,1} \tag{22b}$$

$$g_{o,1} = \frac{f_{o,1} + \sum_{j=3}^{J} f_{o,j} - 0.5 f_{o,1}^2}{g_{o,1}} \tag{22c}$$

For $i \geq 1$

$$h_{i,1} = -\sum_{j=2}^{J} h_{i,j} \tag{22d}$$

$$h_{i,1} = 0.5\left( h_{i,1} + \sum_{l=1}^{i} s_l h_{i-l,1} \right) \tag{22e}$$

$$g_{i,1} = \frac{1}{2g_{o,1}}\left( 2f_{i,1} + 2\sum_{j=3}^{J} f_{i,j} - \sum_{l=1}^{i-1} g_{l,1} g_{i-l,1} + \sum_{l=0}^{i}\left( f_{l,1} f_{i-l,1} - \sum_{m=0}^{l} s_m f_{l-m,1} \right) h_{i-l,1} - \sum_{l=1}^{i} (2s_l + 3c_{l-1}) f_{i-l,2} \right) \tag{23f}$$

4. Check for convergence: i) if reached, go to (c) else go to 2.

(c) Calculate $c_i$ using Eqs. (20).

Calculations based on the numerical procedure described above were performed in double precision arithmetic. A value of 50 was used for the cutoff number $J$, for all values of $i$ considered. The starting values for $(f_{i,1}, g_{i,1})$ were taken to be (1,1) when $i$=0 and (0,0) when $i \geq 1$. Convergence was considered reached when the variation in each of $f_{i,1}$, and $g_{i,1}$ in two consecutive iterations did not exceed $10^{-7}$.

Figures 2-4 show the variation of the velocity components with the vertical coordinate $\zeta$ for different small values of $\beta$. Three-term expansion in powers of $\beta$ has been calculated and the results agree with those obtained by the numerical solution. The figures show the stabilizing effect of the magnetic field. It is seen that as $\beta$ increases the flow becomes more rigid with the velocity components diminishing over most of the domain. The region close to the disk where fast changes take place also contracts in size as $\beta$ grows. In addition, comparison between the figures shows that for the same step of increase in $\beta$, the reduction in the vertical velocity component *H* is higher than that in the radial component *F* and much higher than that in the azimuthal component *G*. This is due to the fact that, the centrifugal effect is the source of the radial motion which is the source of the vertical motion. Hence the reduction in the azimuthal velocity affects the radial velocity which in turn affects the vertical velocity. In addition, Figs. (2) and (3) show that the azimuthal and radial velocity components reach the far field conditions at a finite distance from the disk that decreases with increasing $\beta$. Also, Fig. (4) shows that the vertical velocity component reaches a saturated value at a finite distance from the disk.

Figures 5-7 show the variation of the azimuthal wall shear $\gamma$, radial wall shear $\tau$, and the vertical velocity component at infinity $H_\infty$ in the range $0 \leq \beta \leq 1$. Comparison between the numerical solution, the two-term and the three-term expansions is shown. The figures show that the three solutions are very close to each other for small $\beta$ which proves the validity of the extrapolated numerical solution. It shows also that the two-term expansion is sufficient to obtain accurate solution within that range of $\beta$. As $\beta$ increases differences between the three solutions appear and addition of new terms of the expansions prove necessary. Comparison with the numerical solution shows the validity of the three-term expansion up to values of $\beta$ close to unity. Better accuracy can be obtained by adding new terms using the numerical procedure discussed before. The three solutions are closer to each other in Fig. 5 than they are in Figs. 6 and 7. This can be attributed to the fact that the flow velocity in the azimuthal direction and the corresponding flow variables (as the azimuthal wall shear) do not vary greatly with changes in $\beta$ as the radial and vertical velocities are doing (see Figs. 2-4). It should be noted that the coincidence of the two-term expansion and the extrapolated numerical solution shown in Fig. (5) is obviously accidentally and may be explained by the weak dependence of the azimuthal wall shear on $\beta$. The addition of the third term moves the series solution away from the numerical solution. In Fig. (7), the two-term expansion leads to positive vertical velocity component for values of $\beta$ near unity which is physically unacceptable. Three-term expansion does not face this problem.

Figure 5 shows that the increase in the value of $\beta$ leads to an increase in the magnitude of the azimuthal wall shear $\gamma$. Thus as $\beta$ increases the applied torque required for prescribed steady state velocity distribution is increased. Figure 6 indicates that, the increase in $\beta$ leads to a decrease in the vertical velocity component at infinity. These results are due to the rigidity acquired by the fluid as a result of the influence of the magnetic field.

### 4. Asymptotic solution for large $\beta$

To determine the asymptotic solution of the flow as $\beta$ goes to infinity, we first scale the flow variables with $\beta$ in a suitable way that expresses their correct limiting behavior described above. The fact that *G* assumes a value of unity at the surface irrespective of $\beta$, guarantees that $G = O(\beta^0)$. A quick study of Eq. (6c) shows that for the diffusion term $G''$ to balance the porous force term $\beta G$, the coordinate $\zeta$ would have to be $O(\beta^{-1/2})$. The two corresponding terms $F''$ and $\beta F$ in Eq. (6b) have the same order $O(\beta F)$. However, since the radial motion is driven by the centrifugal effect expressed by $G^2$, it is expected that $\beta F = O(G^2)$, leading to $F = O(\beta^{-1})$. Further, $H = O(\beta^{-3/2})$, for the two terms of the continuity Eq. (6a) to be of the same order. Finally, Eq. (6d) implies $Q = O(\beta^{-1})$.

We introduce new variables $\tilde{\zeta}$, $\tilde{F}$, $\tilde{G}$, $\tilde{H}$, $\tilde{Q}$ defined as follows

$$\zeta = \beta^{-1/2}\tilde{\zeta}, F = \beta^{-1}\tilde{F}, G = \tilde{G}, H = \beta^{-3/2}\tilde{H}, Q = \beta^{-1}\tilde{Q} \tag{24}$$

in terms of which the flow equations and boundary conditions become

$$H' + 2F = 0, \tag{25a}$$

$$F'' + F = -G^2 + \beta^{-2}(HF' + F^2), \tag{25b}$$

$$G'' + G = \beta^{-2}(HG' + 2FG), \tag{25c}$$

$$Q' = H'' - \beta^{-2}HH' - H, \tag{25d}$$

$$\zeta \to \infty : F \to 0, G \to 0, Q \to 0 \tag{26a,b,c}$$

$$F(0) = 0, G(0) = 1, H(0) = 0 \tag{27a,b,c}$$

where the tildes have been dropped. This suggests straightforward expansions for $\tilde{F}$, $\tilde{G}$, $\tilde{H}$, $\tilde{Q}$ that proceed in powers of $\beta^{-2}$. However, these expansions are found to contain secular terms; e.g., the expansion for $\tilde{G}$ is

$$\tilde{G} = \exp(-\zeta) + \beta^{-2}\{\exp(-\zeta) - \exp(-3\zeta) - 4\zeta\exp(-\zeta)\}/24 + ..... \tag{28}$$

Terms of the form $\zeta\exp(-\zeta)$ have secular behavior as $\zeta \to \infty$. To avoid this behavior and to obtain uniformly valid expansions we use the method of strained coordinates (NAYFEH, 1973). We introduce a strained coordinate

$$T = D\zeta \tag{29}$$

and choose the straining function $D$ that eliminates the secular terms. The following two-term expansions are obtained. (The expansions for $\tilde{Q}$ can be deduced from that for $\tilde{H}$).

$$D \approx 1 + \beta^{-2}/6 \tag{30a}$$

$$\tilde{G} \approx E + \beta^{-2}\{E - E^3\}/24 \tag{30b}$$

$$\tilde{F} \approx \{E - E^2\}/3 + \beta^{-2}\{-53E + 10E^2 + 45E^3 + 2E^4\}/1080 \tag{30c}$$

$$\tilde{H} \approx \{-1 + 2E - E^2\}/3 + \beta^{-2}\{127 - 226E + 70E^2 + 30E^3 + E^4\}/1080 \tag{30d}$$

where $E = \exp(-T)$.

Determination of further terms becomes more and more complicated. However, from the first few terms, we can detect general forms for the expansions that can help determine as many terms as we need with the aid of the computer. These forms are

$$D = \sum_{m=1}^{\infty} \beta^{-2(m-1)} d_m \tag{31a}$$

$$\widetilde{G} = \sum_{m=1}^{\infty} \left\{ \beta^{-2(m-1)} \sum_{n=1}^{2m-1} g_{nm} E^n \right\} \tag{31b}$$

$$\widetilde{F} = \sum_{m=1}^{\infty} \left\{ \beta^{-2(m-1)} \sum_{n=1}^{2m} f_{nm} E^n \right\} \tag{31c}$$

$$\widetilde{H} = \sum_{m=1}^{\infty} \left\{ \beta^{-2(m-1)} \sum_{n=0}^{2m} h_{nm} E^n \right\} \tag{31d}$$

with the requirements

$$\sum_{n=1}^{2m-1} g_{nm} = \delta_{1m}, \sum_{n=1}^{2m} f_{nm} = 0, \sum_{n=0}^{2m} h_{nm} = 0 \tag{32a,b,c}$$

where $\delta_{1m}$ is the Kronicker delta. These requirements guarantee the satisfaction of the surface conditions (27). The conditions (26) valid as $T \to \infty$ ($\zeta \to \infty$) are satisfied automatically on account of the exponentials appearing in the expansions. Substituting the expansions (31) in Eqs. (25), and equating the coefficients of like terms, we obtain algebraic equations whose solutions for the *d*, *g*, *f*, *h*'s are given in Appendix A. When expansions (25) are used to calculate *F*, *G*, and *H*, the results are found to coincide, within plotting accuracy, with the numerical solutions for $\beta$ as low as 1.

## CONCLUSIONS

The limiting behaviour of the hydromagnetic flow due to a rotating disk as the magnetic parameter $\beta$ tends to zero or infinity has been established and the expansions of the flow variables have been found to proceed in powers of $\beta$ or in powers of $\beta^{-2}$, respectively. Straining of coordinates was used to remove a secular behavior and lead to a systematic determination of the expansion coefficients. Then, it becomes possible to produce the expansion to any order. Comparison with an exact numerical solution for the governing equations proved the validity of the expansions even for moderate values of $\beta$.

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**Appendix A**

The expressions for the *d*, *g*, *f*, *h*'s are

$$d_1 = 1.0 \tag{A1d}$$

$$g_{11} = 1.0 \tag{A1g}$$

$$f_{11} = 1/3, f_{21} = -1/3 \tag{A1f}$$

$$h_{01} = -1/3, h_{11} = 2/3, h_{21} = -1/3 \tag{A1h}$$

For $m$=2,3,….

$$d_m = \left\{ -\sum_{j_1=2}^{m-1} \left( \sum_{j=1}^{j_1} d_j d_{j_1-j+1} \right) g_{1,m-j_1+1} - \left( \sum_{j_1=2}^{m-1} d_{j_1} d_{m-j_1+1} \right) - \sum_{j_1=1}^{m-1} \left( \sum_{j=1}^{j_1} d_j h_{1,j_1-j+1} \right) g_{1,m-j_1} \right\} / 2 \tag{A2d}$$

$$g_{nm} = \{-\sum_{j=2}^{m-1}\left(\sum_{k=1}^{j} d_k d_{j-k+1}\right) g_{n,m-j+1} n^2 + 2\sum_{j=1}^{m-1}\left(\sum_{j_2=1}^{2j} \sum_{\substack{j_3=1 \\ j_2+j_3=n}}^{2m-2j-1} f_{j_2,j}\, g_{j_3,m-j}\right) +$$

$$\sum_{j=1}^{m-1} d_{m-j} \sum_{j_4=1}^{j} \sum_{j_2=1}^{2j_4+1} \sum_{\substack{j_3=1 \\ j_2+j_3-1=n}}^{2j-2j_4+1} h_{j_2,j_4}\, g_{j_3,j-j_4+1}\left(-j_3\right)\}/(n^2-1)$$

$n = 2 \to 2m - 1$ (A2g)

$$f_{nm} = \{-\sum_{j=2}^{m}\left(\sum_{k=1}^{j} d_k d_{j-k+1}\right) f_{n,m-j+1} n^2 - \sum_{j=1}^{m} \sum_{j_2=1}^{2j-1} \sum_{\substack{j_3=1 \\ j_2+j_3=n}}^{2m-2j+1} g_{j_2,j}\, g_{j_3,m-j+1} +$$

$$\sum_{j=1}^{m-1} \sum_{j_2=1}^{2j} \sum_{\substack{j_3=1 \\ j_2+j_3=n}}^{2m-2j} f_{j_2,j}\, f_{j_3,m-j} +$$

$$\sum_{j=1}^{m-1} \sum_{j_4=1}^{j} \sum_{j_2=1}^{2j_4+1} \sum_{\substack{j_3=1 \\ j_2+j_3-1=n}}^{2j-2j_4+2} h_{j_2,j_4}\, f_{j_3,j-j_4+1}(-j_3)\, d_{m-j}\}/(n^2-1)$$

$n = 2 \to 2m$ (A2f)

$$h_{nm} = \{\sum_{j=2}^{m} d_j d_{n,m-j+1}(-n+1) + 2 f_{n-1,m}\}/(n-1)$$

$n = 1 \to 2m$ (A2h)

$$g_{1m} = -\sum_{n=2}^{2m-1} g_{nm} \qquad \text{(A3g)}$$

$$f_{1m} = -\sum_{n=2}^{2m} f_{nm} \qquad \text{(A3f)}$$

$$h_{0m} = -\sum_{n=1}^{2m} h_{nm} \qquad \text{(A3h)}$$

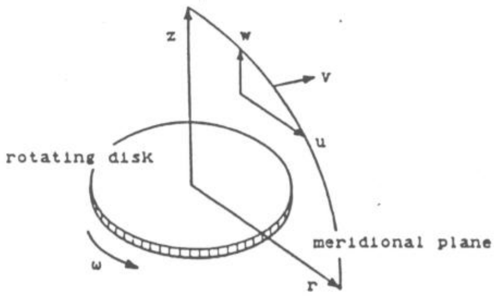


Fig. 1: Flow Configuration

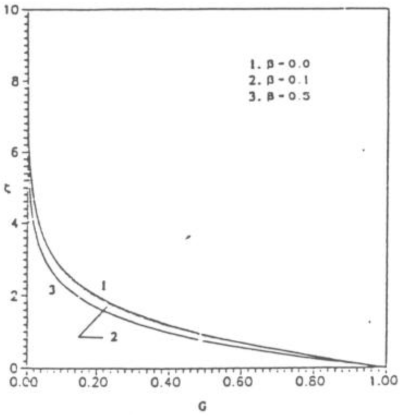


Fig. 2: Azimuthal velocity profile for different values of β

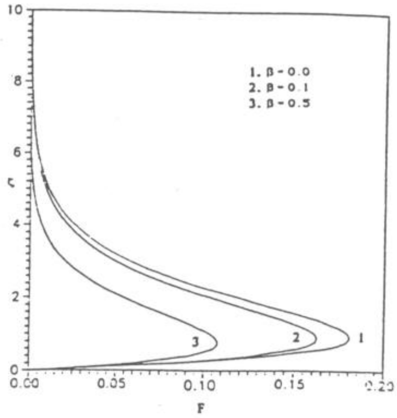


Fig. 3: Radial velocity profile for different values of β

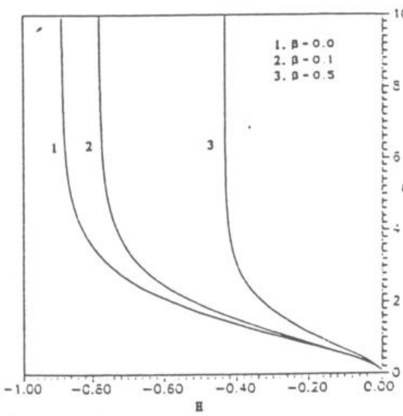


Fig. 4: Vertical velocity profile for different values of β

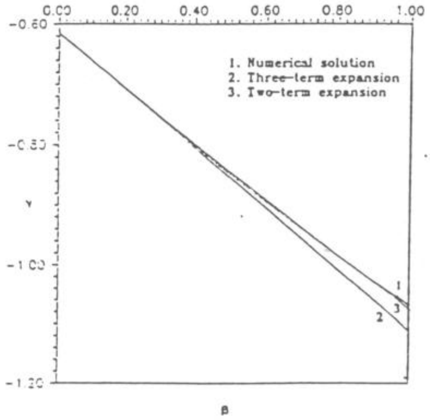


Fig. 5: Variation of the azimuthal wall shear with β

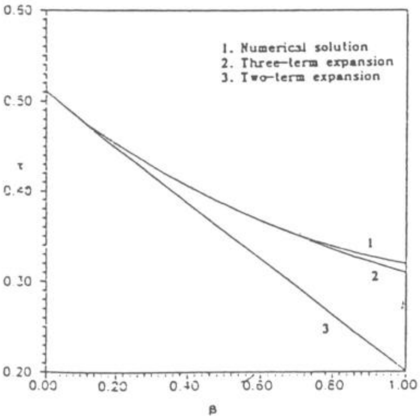


Fig. 6: Variation of the radial wall shear with β

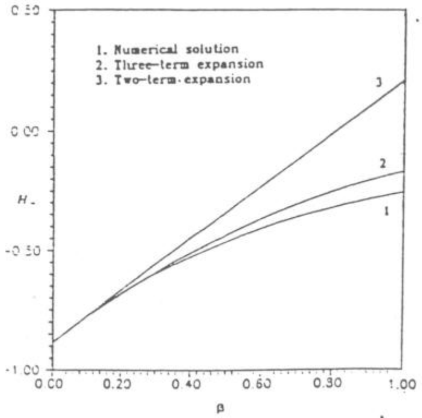


Fig. 7: Variation of the vertical velocity at infinity with β