Kragujevac J. Sci. 25 (2003) 31-40.

IMPROVED NUMERICAL SOLUTIONS TO PLASMA LOADED HELICAL WAVEGUIDE

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(Received March 14, 2003)

ABSTRACT. The behavior of axially-symmetric waves traveling on plasma filled screened helical coil is comprehensively studied in [1], where all analytical expressions are derived in the full-wave theory, but the diagrams are computed in the quasistatic approximation. In this paper we have overcome this limitation by means of the fixed-point method. We have proved that the first-order approximant is sufficient to improve defects which are known as immanent to the quasistatic approximation. There is no need for use of second or higher order approximants, because the relative difference between any two adjacent (higher than fist) approximants is practically negligible in analyzed domain of wavenumbers. The improvements in dispersion graphs are essential for $\beta a < 0.5$. The exact solution experiences a threshold which depends on plasma density. The first-approximant can reproduce this feature. For $\beta a > 1$ the relative difference cannot exceed 5% (in most cases an unessential correction). We have developed programs, relying on the fixed-point method which may also be apply in more complex numerical manipulations. A couple of numbers (X, Y) could be deduced (with a required error) from the dispersion equation in closed form and automatically used in subsequent steps of computations.

1. INTRODUCTION

The problem of the propagation of electromagnetic waves along a wire coil surrounded by a highly conductive screen has been treated in several papers [e.g. 5, 6] and with electron plasma present in the coil interior thoroughly investigated in [1]. The author succeeded in obtaining a dispersion relation in finite form. This dispersion relation was proved as correct in limiting cases a) when the coil is removed and b) when the plasma is non-existent. The wave on such a guiding structure is always faster then the plasma or coil wave taken separately.

The numerical results in [1] are obtained under the reasonable approximation which starts from the fact that guided waves are much slower then the speed of light (so-called the quasistatic approximation). We have argued in the paper [4] that the fixed-point method often could be with benefit applied in many guided wave propagation problems. In this connection, the fixed-point method was completely tested through-out the all captures of the book [2]. We have developed software packets which avoid any starting approximations and successfully calculate expressions in its full-electromagnetic formulations. Our task is to show here that the problem of electromagnetic waves on a helix with a plasma core could be an appropriate example that the existing results based on the quasistatic approximation can be improved by means of the fixed-point technique.

We plan to arrange this paper as follows. In Section 2 we repeat (in an abridged form) the derivation of main expressions which will be later analyzed. In Section 3 the reader can find the necessary review on the fixed point method, in an extent which seems to be here appropriate. Section 4 deals with numerical treatment of expressions derived in Section 2. Finally, in Section 5 we give a brief conclusion with comparisons and comments.

2. SEARCHING FOR DISPERSION RELATION

2.1 *Structure of the line*

The presentation in this Section will follow the course adopted in [1], with the exception of several designations which shall be on time explained. In Fig. 1 we see the model of the physical guiding structure under consideration.

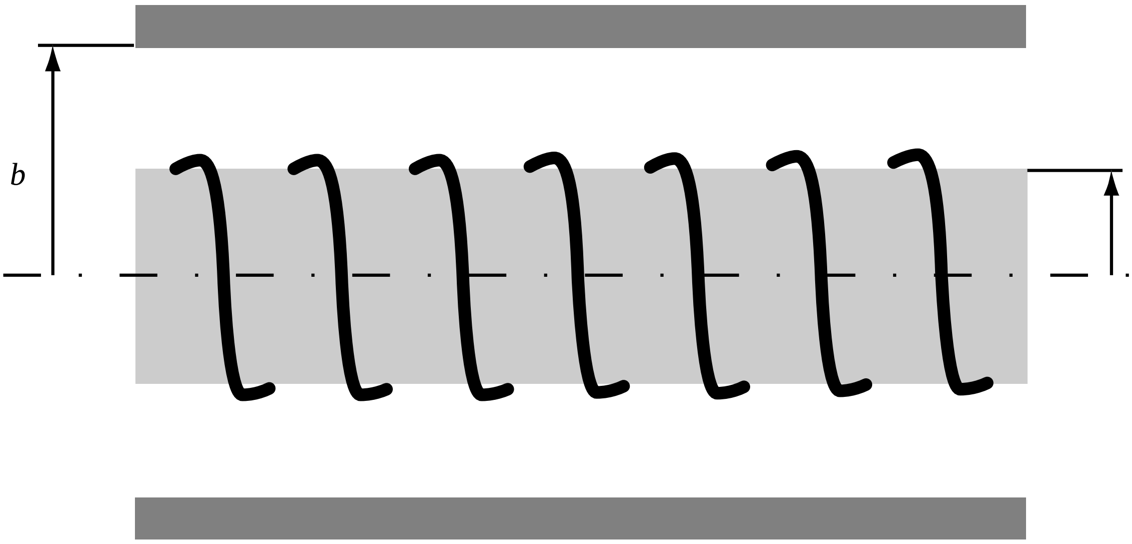

Fig. 1. Guiding structure under consideration; cylindrical metal sheet at $r = b$; wire coil at $r = a$; coil interior filled with homogeneous cold plasma; region between coil and cylinder: classical dielectric

The central part of the line consists of a coil, the interior of which is filled with homogeneous plasma. The coil is surrounded with a metal cylinder. Of course, one could imagine plasma is suspended in a glass tube; but we shall treat the glass wall as a very thin one. The region between coil and cylinder is a dielectric characterized with permittivity ε_{r} (in the case of a coil in air, $\varepsilon_{\mathrm{r}} \approx 1$). The coil we imagine as a tape helix which is a infinitely thin anisotropic sheet of current flowing obliquely to the coil axis, on the $r = a$ cylindrical surface. The radius of the conductive screen is $r = b$.

2.2 *Features of the field*

Now we shall consider the character of the field which could develop in these circumstances. We restrict ourselves only to the treatment of an axially symmetric mode, which is well known [4] in the theory of electron space charge waves (often called simply as $n = 0$ mode).

The adequate coordinate system here is a cylindrical system (r, φ, z); the z -axis coincides with the axis of the guide, whereas r and φ are corresponding radius and angle in the perpendicular plane. Therefore, the components of the electric field vector $\vec{E}$ are ($E_{\mathrm{r}}, E_{\varphi}, E_{z}$)

and the magnetic field vector $\vec{H}$ are ($H_{\mathrm{r}}, H_{\varphi}, H_{\mathrm{z}}$). So-called H (or TE) field is composed of the group $E_{\varphi}, H_{\mathrm{r}}, H_{\mathrm{z}}$ and an E (or TM) field has the components $H_{\varphi}, E_{\mathrm{r}}, E_{\mathrm{z}}$.

A) The H field in the plasma, as it emerges from Maxwell's equations, is as follows:

$$E_{\varphi} = C_1 I_1 \left(h_1 r \right) \tag{1}$$

$$H_{\mathrm{r}} = -\frac{k}{\omega \mu_0} C_1 I_1 \left(h_1 r \right) \tag{2}$$

$$H_{\mathrm{z}} = -\frac{i h_1}{\omega \mu_0} C_1 I_0 \left(h_1 r \right) \tag{3}$$

Here, I_{ν} stands for the modified Bessel function of the first kind of order ν. Our abbreviation h_1 means

$$h_1 = \sqrt{k^2 - \frac{\omega^2 \varepsilon_{\mathrm{p}}}{c_0^2}} \tag{4}$$

Above, k is the wavenumber, ω is the radian frequency of a continuous wave signal propagating along the guide, ω_{p} is the radian plasma frequency and c_0 is the velocity of light. The plasma permittivity is, within the frame of a cold, homogeneous ionized medium

$$\varepsilon_{\mathrm{p}} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}. \tag{5}$$

In the space between coil and metal sheet the expressions are of the form

$$E_{\varphi} = C_2 K_1 \left(h_2 r \right) + C_3 I_1 \left(h_2 r \right) \tag{6}$$

$$H_{\mathrm{r}} = -\frac{k}{\omega \mu_0} \left[C_2 K_1 \left(h_2 r \right) + C_3 I_1 \left(h_2 r \right) \right] \tag{7}$$

$$H_{\mathrm{z}} = \frac{i h_1}{\omega \mu_0} \left[-C_2 K_0 \left(h_2 r \right) + C_3 I_0 \left(h_2 r \right) \right] \tag{8}$$

B) Let us now quote expressions valid for E field. Within the coil the components are:

$$H_{\varphi} = C_1^{\mathrm{E}} I_1 \left(h_1 r \right), \tag{9}$$

$$E_{\mathrm{r}} = -\frac{ik}{\omega \varepsilon_0} C_1^{\mathrm{E}} I_1 \left(h_1 r \right), \tag{10}$$

$$E_{\mathrm{z}} = \frac{i h_1}{\omega \varepsilon_0} C_1^{\mathrm{E}} I_0 \left(h_1 r \right). \tag{11}$$

The E field in the space between coil and lining is as follows:

$$H_{\varphi} = C_2^{\mathrm{E}} K_1 \left(h_2 r \right) + C_3^{\mathrm{E}} I_1 \left(h_2 r \right), \tag{12}$$

$$E_{\mathrm{r}} = \frac{k}{\omega\,\varepsilon_0\varepsilon_{\mathrm{r}}}\left[C_2^{\mathrm{E}}K_1\left(h_2r\right)+C_3^{\mathrm{E}}I_1\left(h_2r\right)\right], \tag{13}$$

$$E_{\mathrm{z}} = -\frac{ih_2}{\omega\,\varepsilon_0\varepsilon_{\mathrm{r}}}\left[-C_2^{\mathrm{E}}K_0\left(h_2r\right)+C_3^{\mathrm{E}}I_0\left(h_2r\right)\right]. \tag{14}$$

Here K_{v} stands for the modified Bessel functions of the second kind of order v and

$$h_2 = \sqrt{k^2 - \frac{\omega^2\varepsilon_{\mathrm{r}}}{c_0^2}}\,. \tag{15}$$

It is known that in the limit of large wavenumbers k the field is concentrated close to the plasma boundary (and coil too); at small wavenumbers, on the contrary, the wave fills the whole of the guide space.

2.3 *Boundary conditions*

Let us impose the boundary conditions on the obtained components of field. The tangential components of the electric field vanish at the surface of the highly conductive screen, i.e. for $r = b$. At the surface separating plasma from outer dielectric continuity of the tangential electric fields must be satisfied. At the same time, there the tangential components of the magnetic field are discontinuous, because a surface current flows over the cylinder at $r = a$.

On the other hand, the field along the wire must vanish (the helix wire is made of a highly conductive material). The total electric field must be normal to the wire. We accept therefore a fictitious current sheet model and demand the condition

$$\tan\Psi = -\frac{E_{\varphi}\left(a\right)}{E_{\mathrm{z}}\left(a\right)} = \frac{p}{2\pi a}, \tag{16}$$

where p is the pitch and Ψ is the pitch angle of the helix. Through this condition, as seen, waves on our structure become hybrid mode waves, containing all field components.

2.4 *Dispersion relation of waves on a plasma loaded screened coil*

Using the mentioned boundary conditions we can eliminate the constants appearing in the field expressions ($C_1, C_2, C_3, C_1^{\mathrm{E}}, C_2^{\mathrm{E}}, C_2^{\mathrm{E}}$). This treatment gives the dispersion relation of waves on the composite plasma-coil-dielectric-screen guide in the form

$$\left(\frac{\omega a}{c_0}\right)^2 = P^2\left(\frac{\omega_{\mathrm{p}}a}{c_0}\right)^2 + H^2\tan^2\Psi \tag{17}$$

The normalized radian wave frequency $\omega a / c_0$ and the normalized radian plasma frequency $\omega_{\mathrm{p}}a / c_0$ are mutually connected by means of two functions of frequency and wavenumber P and H which are defined as follows:

$$P = \left[\frac{E\left(h_1\right)}{D\left(h_1, h_2\right)}\right]^{1/2}; \tag{18}$$

$$H = \left\{ \frac{1}{D(h_1, h_2)} \left[\frac{1}{E(h_1)} + F(h_2) \right] \right\}^{0.5} . \tag{19}$$

Here we have abbreviated

$$E(h_1) = \frac{1}{h_1 a} \frac{I_1(h_1 a)}{I_0(h_1 a)}, \tag{20}$$

$$D(h_1, h_2) = E(h_1) + \varepsilon_r G(h_2). \tag{21}$$

In these expressions appear the functions $F(h_2)$ and $G(h_2)$ defined by

$$F(h_2) = h_2 a \frac{I_1(h_2 b) K_0(h_2 a) + I_0(h_2 a) K_1(h_2 b)}{I_1(h_2 b) K_1(h_2 a) - I_1(h_2 a) K_1(h_2 b)}, \tag{22}$$

$$G(h_2) = \frac{1}{h_2 a} \frac{I_0(h_2 b) K_1(h_2 a) + I_1(h_2 a) K_0(h_2 b)}{I_0(h_2 b) K_0(h_2 a) - I_0(h_2 a) K_0(h_2 b)}. \tag{23}$$

The correctness of the equation (17) can be tested in two limited cases. a) For small helix pitch $\tan\Psi \approx 0$ and $\omega/\omega_p = P$, which coincides with the expression for the dispersion relation for plasma surface waves on a plasma of radius a within a shield of radius b [4]. b) When no plasma is present in the coil must be $\omega_p = 0$ and $(\omega a / c_0)\cot\Psi = H$, which is really the dispersion relation of waves on a wire coil originally published in [5].

In the paper [1] the author computed the dispersion relation in the quasistatic approximation which involves the inequality $k \gg \omega/c_0$. As we have already stressed, our task is to compute the dispersion relation applying the fixed-point method, without the degradation of eq. (17) to the quasistatic level. According to plan we shall first in brief explain the mean of the method.

3. FIXED-POINT METHOD

As we explained in the paper [4], dispersion relations of surface waves in various combinations of media containing plasma may be put in a form which immediately refers to the existence of a guiding surface:

$$-\varepsilon_p = \varepsilon_r \, f\left[X, Y(X)\right]. \tag{24}$$

This equation could simultaneously serve as a definition of the corresponding *form factor* f which we shall name the *surface function* of a guiding structure. The surface function satisfies the next condition:

$$\lim_{X \to \infty} f\left[X, Y(X)\right] = 1 . \tag{25}$$

The wavenumber β appears in the normalized quantity $X = \beta a$ (a represent a suitable radius or another corresponding normalizing length). The quantity Y represents the angular wave frequency normalized with the angular plasma frequency, i.e. $Y = \omega\, \omega_p^{-1}$. In the important case that the plasma permittivity is as in eq. (5), dielectrics are homogeneous and mutually in contact in accordance to the model of a sharp boundary, the dispersion relation takes the form

$$Y = \frac{1}{\sqrt{1+\varepsilon_{\mathrm{r}}\ f}}. \tag{26}$$

Let us $Y_0(X)$ be the zero approximant (derived from the equivalent dielectric method [3], or merely in the quasistatic treatment). Putting that value in the right-hand-side of eq. (26) we get formally the first-approximant solution:

$$Y_1 = f\left(X, Y_0(X)\right). \tag{27}$$

Continuing now this iterative procedure, we can use the first-order approximant as a new starting result in the right-hand-side of (26) and deduce Y_2, the second-order approximant:

$$Y_2 = f\left(X, f\left(X, Y_1(X)\right)\right). \tag{28}$$

In the same manner one could get any approximant of a order $i > 2$; the expression is

$$Y_i = f\left(X, f\left(X, Y_{i-1}(X)\right)\right). \tag{29}$$

We proved in [4] that this iteration converges very fast and in most standard cases the solution Y_1 is quit satisfactorily. Now we are ready to apply this method in the case of waves on plasma loaded helix.

4. IMPROVED NUMERICAL RESULTS

4.1 *Formula preparing*

Starting from the equation (17) of our section 2 one can (in a somewhat tiresome but otherwise straightforward mathematical manipulations) obtain the following expression:

$$Y = \frac{1}{\sqrt{1+\varepsilon_{\mathrm{r}}\ f_{\mathrm{coil}}}}, \tag{30}$$

Where the loaded helix surface function (in a sense of the definition (24)) reads

$$f_{\mathrm{coil}} = \frac{D_1 p_0^2 \cot^2\Psi - G_1}{\varepsilon_{\mathrm{r}} p_0^2 \cot^2\Psi \dfrac{I_1(h_1 a)}{h_1 a\, I_0(h_1 a)}}. \tag{31}$$

The understanding of this expression goes over the next set of subsequent definitions:

$$D_1 = \frac{\varepsilon_{\mathrm{r}}}{h_2 a} \frac{\dfrac{K_1(h_2 a)}{K_0(h_2 b)} + \dfrac{I_1(h_2 a)}{I_0(h_2 b)}}{\dfrac{K_0(h_2 a)}{K_0(h_2 b)} - \dfrac{I_0(h_2 a)}{I_0(h_2 b)}}; \tag{32}$$

$$G_1 = \frac{h_1 a\, I_0(h_1 a)}{I_1(h_1 a)} + h_2 a \frac{\dfrac{K_0(h_2 a)}{K_1(h_2 b)} + \dfrac{I_0(h_2 a)}{I_1(h_2 b)}}{\dfrac{K_1(h_2 a)}{K_1(h_2 b)} - \dfrac{I_1(h_2 a)}{I_1(h_2 b)}}; \quad (33)$$

$$X = \beta a; \quad Y = \frac{\omega}{\omega_{\mathrm{p}}}; \quad p_0 = \frac{\omega a}{c_0}; \quad d = \frac{b}{a}; \quad (34)$$

$$h_1 a = \sqrt{X^2 - p_0^2 \varepsilon_{\mathrm{p}}}; \quad (35)$$

$$h_2 a = \sqrt{X^2 - p_0^2 \varepsilon_{\mathrm{r}}}; \quad (36)$$

$$h_2 b = d\, h_2 a; \quad \varepsilon_{\mathrm{p}} = 1 - \frac{1}{Y^2}. \quad (37)$$

The zero or quasistatic approximation could be easily deduced from eq. (30) because it turns into the explicit form $Y = F(X)$ after the reduction $h_1 a \approx h_2 a \approx X = \beta a$. A search for the next and higher approximants is described in section 3.

4.2 *'Mathematica 4.0' graphics*

We have developed a program which is capable, in frame of *Mathematica 4.0* package, to efficiently compute and plot the main features of the above dispersion relation. A listing of the program is delivered as an Appendix at the end of this paper. In this section let us see only figures we obtained in various cases when running the program.

Fig. 2 shows the dispersion relation, eq. (17), for $p_0 = 0.1$, $a = 5\,\mathrm{cm}$, $d = 3$ and $\varepsilon_{\mathrm{r}} = 4.8$ (Pyrex glass). The pitch of the helix is chosen to be $p = 0.001$, giving $\cot\Psi = 31.416$. The upper curve is the quasistatic solution and the lower curve is the first approximant. We see that the frequency monotonically grows in the range $X \in (0, 2)$. The fixed-point first approximant is always below the zero approximant. The difference $Y_0 - Y_1$ diminishes when the wavenumber grows. Below the value $Y \approx 0.05$ the quasistatic solution obviously failed; the exact solution does not run from the origin – the curve has a threshold point. The Y_1 solution regularly tends toward a finite value in the limit $Y \to 0$.

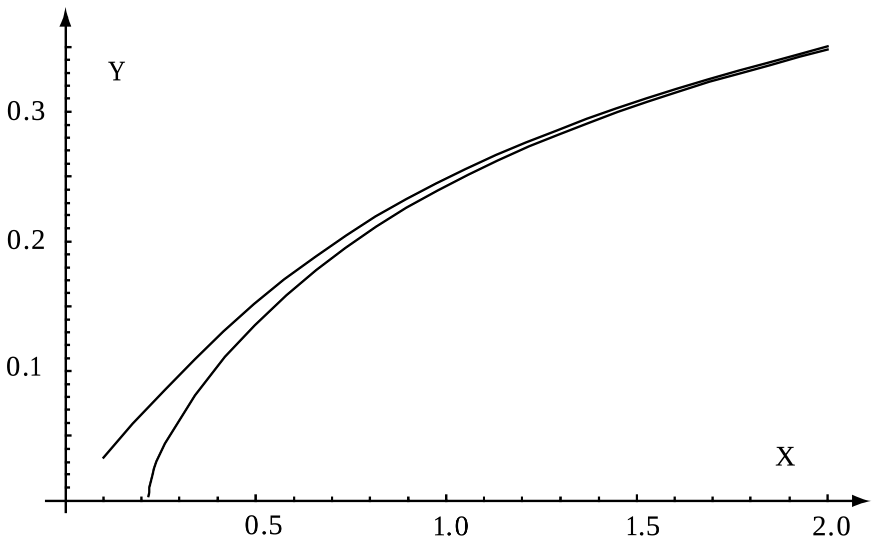

Fig. 2. Comparison of two solutions; upper curve – quasistatic solution; lower curve – fixed-point first approximant.
Parameters: $p_0 = 0.1$, $a = 5$ cm, $d = 3$, $\varepsilon_r = 4.8$, $p = 0.001$ ($\cot\Psi = 31.416$)

Fig. 3 shows the dispersion relation when the parameter d vary (i.e. for several b/a quotients). The sensitivity $s = \delta Y / \delta d$ could reach quit large values. For a given wavenumber the frequency goes down when the quotient falls. All curves start from the same threshold. The six curves are with $d = 10, 3, 2, 1.5, 1.2$ and 1, respectively (up to down direction). The parameters have the same values as in Fig. 2.

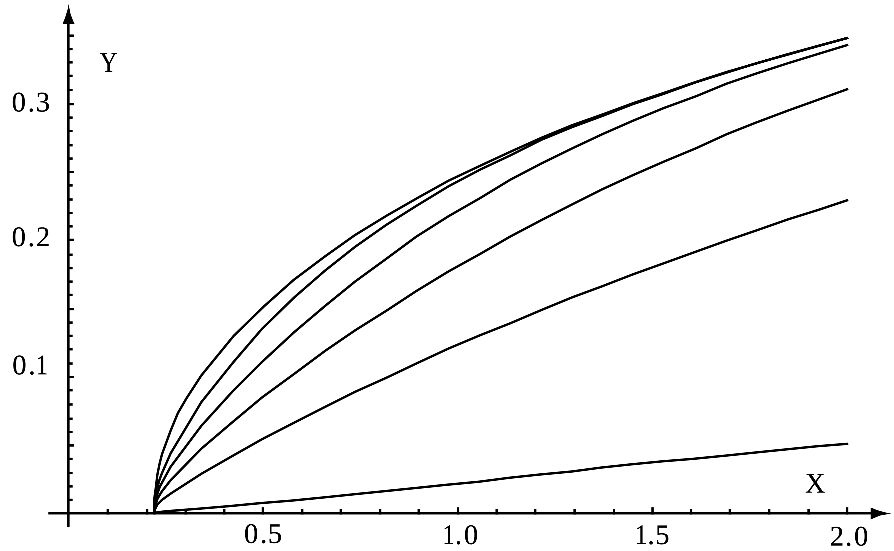

Fig. 3. Dispersion curve and its d-sensitivity (fixed-point first approximant); up to down: $d = 10, 3, 2, 1.5, 1.2, 1.01$; Parameters: $p_0 = 0.1$, $a = 5$ cm, $\varepsilon_r = 4.8$, $p = 0.001$ ($\cot\Psi = 31.416$)

Fig. 4 clears up the impact of the plasma density factor p_0 on the dispersion equation. Four curves are presented and these are for $p_0 = 0.1, 0.08, 0.06, 0.04$ (from below to above). As expected, the curves have not the same threshold, and exactly the smallest threshold belongs to the smallest plasma density factor. In the central part of the diagram the dependence $Y(p_0)$ is but little. However, in the region X □ 2 we see huge differences; for small p_0 the frequency takes a fast raising trend. Once again, the other parameters in this figure are the same as in Fig 2.

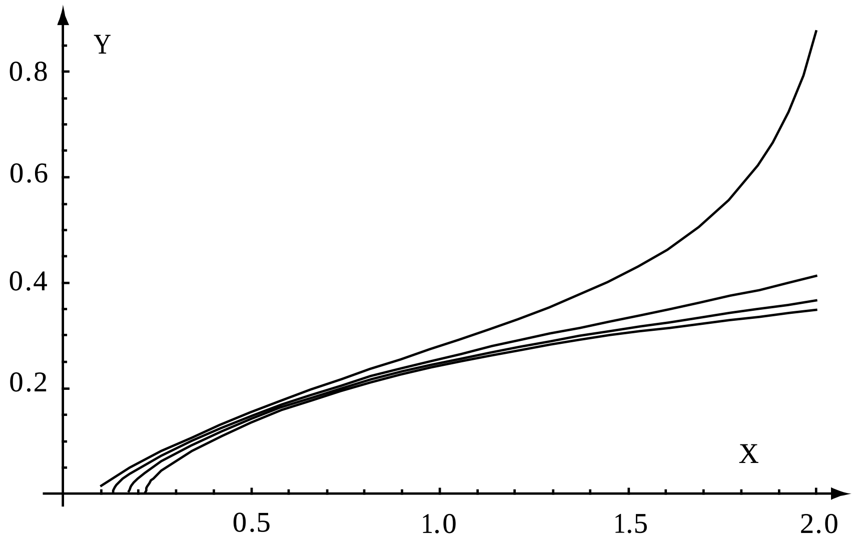

Fig. 4. Impact of plasma density factor on dispersion curve (fixed-point first approximant); $p_0 = 0.1$, 0.08, 0.06, 0.04 (from above to below). Parameters: $d = 3$, $a = 5$ cm, $\varepsilon_r = 4.8$, $p = 0.001$ ($\cot\Psi = 31.416$)

5. CONCLUSIONS AND COMMENTS

The behavior of axially-symmetric waves traveling on plasma filled screened helical coil is studied in [1], where all analytical expressions are derived in the full-wave theory. However, the diagrams are computed in the quasistatic approximation. In this paper we have overcome this limitation by means of the fixed-point method. The method secures a fast-convergent procedure [4]. We have proved here that the first-order approximant Y_1 is sufficient to improve defects which are known as immanent to the quasistatic approximation. There is no need for use of second (Y_2) or higher order approximants. The relative error $(Y_1 - Y_2)/Y_1$ is practically negligible in analyzed domains of wavenumbers.

The improvements are essential for $\beta a < 0.5$. The exact solution experiences a threshold which is dependent on plasma density parameter p_0. The approximant Y_1 can reproduce this feature. The threshold tends to the diagrams origin when $p_0 \to 0$. Consequently, the quasistatic solution has an intrinsic error for $X \to 0$. For $\beta a > 1$ the relative difference $(Y_0 - Y_1)/Y_0$ cannot exceed 5% which is in most cases an unessential correction.

The programs we have developed and here activated, relying on the fixed-point method, may also be apply in more complex numerical manipulations. A couple of numbers (X, Y) could be deduced (with a required error) from the dispersion equation in closed form and automatically used in subsequent steps of computations.

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APPENDIX

Fixed-point method program. General listing features

```
Remove["Global`*"]
xx = x*x; yy = y*y; z = d*x; pp = par*par; ep = 1. - 1./yy;

I0x = BesselI[0, x]; I1x = BesselI[1, x];
K0x = BesselK[0, x]; K1x = BesselK[1, x];
I0z = BesselI[0, z]; I1z = BesselI[1, z];
K0z = BesselK[0, z]; K1z = BesselK[1, z];
Dgore = K1x/K0z + I1x/I0z; Ddole = K0x/K0z - I0x/I0z;
De = (eg/x)*(Dgore/Ddole); Ggore = K0x/K1z + I0x/I1z;
Gdole = K1x/K1z - I1x/I1z;
Ge = x*(I0x/I1x) + x*(Ggore/Gdole);
y0gore = Ge - pp*ctgpsi*ctgpsi*De; y0dole = pp*ctgpsi*ctgpsi*(1./x)*(I1x/I0x);
y0 = Sqrt[1./(1. - y0gore/y0dole)];

h1a = Sqrt[xx - pp*ep]; h2a = Sqrt[xx - pp*eg]; h2b = d*h2a;
I0h1a = BesselI[0, h1a]; I1h1a = BesselI[1, h1a]; I0h2a = BesselI[0, h2a];
I1h2a = BesselI[1, h2a]; K0h2a = BesselK[0, h2a]; K1h2a = BesselK[1, h2a];
K1h2b = BesselK[1, h2b]; K0h2b = BesselK[0, h2b]; I1h2b = BesselI[1, h2b];
I0h2b = BesselI[0, h2b];
Dgore1 = K1h2a/K0h2b + I1h2a/I0h2b; Ddole1 = K0h2a/K0h2b - I0h2a/I0h2b;
De1 = (eg/h2a)*(Dgore1/Ddole1);
Ggore1 = K0h2a/K1h2b + I0h2a/I1h2b; Gdole1 = K1h2a/K1h2b - I1h2a/I1h2b;
Ge1 = h1a*(I0h1a/I1h1a) + h2a*(Ggore1/Gdole1);
y1gore = Ge1 - pp*ctgpsi*ctgpsi*De1; y1dole =
  pp*ctgpsi*ctgpsi*(1./h1a)*(I1h1a/I0h1a);
y = Sqrt[1./(1. - y1gore/y1dole)];
f = Sqrt[1./(1. - y1gore/y1dole)];
Y[y_] := f;

a = 0.005; korak = 0.001; par = 0.02; d = 3.; eg = 4.8; ctgpsi = 2.*Pi*a/korak;
x = 0.1; Label[ovde];
yaN = FixedPoint[Y, y0, N];

Print[{x, yaN}];
x += 0.01; If[x < 1, Goto[ovde]]
```