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ENERGY OF SOME BIPARTITE CLUSTER GRAPHS

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ABSTRACT. The energy of a graph G is defined as the sum of the absolute values of the eigenvalues of G. The graphs with large number of edges are referred as cluster graphs. In this paper we consider the energy of graphs obtained from complete bipartite graph by deleting the edges.

INTRODUCTION

The graphs with large number of edges are viewed as graph representations of inorganic clusters in chemistry, so-called cluster graphs [5]. We call the bipartite graphs with large number of edges as bipartite cluster graphs. In this paper we consider the spectra and energy of some bipartite cluster graphs.

Let G be a simple undirected graph without loops and multiple edges having p vertices and q edges. If the vertices of G are labeled as $v_1, v_2, \ldots, v_p$ then its adjacency matrix A(G) is defined as A(G) = $[a_{ij}]$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The characteristic polynomial of G is defined as $\phi(G : \lambda) = \det(\lambda I - A(G))$, where I is unit matrix of order p. The roots of the equation $\phi(G : \lambda) = 0$ denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G and their collection is called the spectrum of G [2]. The energy of G is defined [3] as $E(G) = |\lambda_1| + |\lambda_2| + \ldots + |\lambda_p|$. It is a generalization of a formula valid for the total π-electron energy of a conjugated hydrocarbon as calculated with the Huckel molecular orbital (HMO) method in quantum chemistry [4].

SOME BIPARTITE CLUSTER GRAPHS

<u>DEFINITION 1:</u> Let e_i, $i = 1, 2, \ldots, k$, $1 \le k \le \min\{m, n\}$, be independent edges of the complete bipartite graph $K_{m,n}$, $m, n \ge 1$. The graph $Ka_{m,n}(k)$ is obtained by deleting e_i, $i = 1, 2, \ldots, k$ from $K_{m,n}$. In addition

$Ka_{m,n}(0) \cong K_{m,n}$.

<u>DEFINITION 2:</u> Let $K_{r,s}$ be the complete bipartite graph and let it be a subgraph of the complete bipartite graph $K_{m,n}$, $1 \le r \le m$, $1 \le s \le n$ and $m, n \ge 1$. The graph $Kb_{m,n}(r,s)$ is obtained by deleting the edges of $K_{r,s}$ from $K_{m,n}$. In addition $Kb_{m,n}(0,0) \cong Kb_{m,n}(r,0) \cong Kb_{m,n}(0,s) \cong K_{m,n}$.

<u>LEMMA 1</u> [2, p. 62]**:**

If M is a nonsingular square matrix then we have

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M|\,|Q - PM^{-1}N| \qquad \square$$

<u>THEOREM 2:</u>

For $m, n \ge 1$ and $0 \le k \le \min\{m, n\}$,

$$\phi(Ka_{m,n}(k) : \lambda) = \lambda^{m+n-2k-2}(\lambda^2 - 1)^{k-1}[\lambda^4 - (m - 2k + 1)\lambda^2 + (m-k)(n-k)]. \qquad (2.1)$$

PROOF: Without loss of generality we partition the vertex set of the complete bipartite graph $K_{m,n}$ into two disjoint sets $A = \{u_1, u_2, \ldots, u_m\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ such that no two vertices in either sets are adjacent to each other. Assume that the independent edges e_i connect the vertices u_i and v_i, $i = 1, 2, \ldots, k$. Then the characteristic polynomial of $Ka_{m,n}(k)$ is the determinant (2.2).

$$\begin{array}{c|cccccccccccccccccc|}
 & u_1 & & & & u_k & u_{k+1} & & u_m & v_1 & & & & v_k & v_{k+1} & & & v_n \\
u_1 & \lambda & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 & -1 & -1 & \ldots & -1 \\
 & 0 & \lambda & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & 0 & -1 & \ldots & -1 & -1 & -1 & \ldots & -1 \\
 & 0 & 0 & \lambda & \ldots & 0 & 0 & \ldots & 0 & -1 & -1 & 0 & \ldots & -1 & -1 & -1 & \ldots & -1 \\
 & \vdots & & & \vdots & & & \vdots & & & & & \vdots & & & & \vdots & \\
u_k & 0 & 0 & 0 & \ldots & \lambda & 0 & \ldots & 0 & -1 & -1 & -1 & \ldots & 0 & -1 & -1 & \ldots & -1 \\
u_{k+1} & 0 & 0 & 0 & \ldots & 0 & \lambda & \ldots & 0 & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & \ldots & -1 \\
 & \vdots & & & \vdots & & & \vdots & & & & & \vdots & & & & \vdots & \\
u_m & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & \lambda & -1 & -1 & -1 & \ldots & -1 & -1 & -1 & \ldots & -1 \\
v_1 & 0 & -1 & -1 & \ldots & -1 & -1 & \ldots & -1 & \lambda & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
 & -1 & 0 & -1 & \ldots & -1 & -1 & \ldots & -1 & 0 & \lambda & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
 & -1 & -1 & 0 & \ldots & -1 & -1 & \ldots & -1 & 0 & 0 & \lambda & \ldots & 0 & 0 & 0 & \ldots & 0 \\
 & \vdots & & & \vdots & & & \vdots & & & & & \vdots & & & & \vdots & \\
v_k & -1 & -1 & -1 & \ldots & 0 & -1 & \ldots & -1 & 0 & 0 & 0 & \ldots & \lambda & 0 & 0 & \ldots & 0 \\
v_{k+1} & -1 & -1 & -1 & \ldots & -1 & -1 & \ldots & -1 & 0 & 0 & 0 & \ldots & 0 & \lambda & 0 & \ldots & 0 \\
 & -1 & -1 & -1 & \ldots & -1 & -1 & \ldots & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & \lambda & \ldots & 0 \\
 & \vdots & & & \vdots & & & \vdots & & & & & \vdots & & & & \vdots & \\
v_n & -1 & -1 & -1 & \ldots & -1 & -1 & \ldots & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \lambda
\end{array} \qquad (2.2)$$

$$= \begin{vmatrix} \lambda I_m & N^T \\ & \\ N & \lambda I_n \end{vmatrix} \qquad (2.3)$$

Where

$$N = \begin{bmatrix} 0 & -1 & -1 \ldots -1 & -1 \ldots -1 \\ -1 & 0 & -1 \ldots -1 & -1 \ldots -1 \\ -1 & -1 & 0 \ldots -1 & -1 \ldots -1 \\ . & & . & . \\ . & & . & . \\ . & & . & . \\ -1 & -1 & -1 \ldots \ 0 & -1 \ldots -1 \\ -1 & -1 & -1 \ldots -1 & -1 \ldots -1 \\ -1 & -1 & -1 \ldots -1 & -1 \ldots -1 \\ . & & . & . \\ . & & . & . \\ . & & . & . \\ -1 & -1 & -1 \ldots -1 & -1 \ldots -1 \end{bmatrix}$$

is a matrix of order $n \times m$ in which the last zero appears in the intersection of k^{th} row and k^{th} column and N^T is the transpose of N.

Applying Lemma 1 [2] to the expression (2.3), we get

$$\lambda^m \left| \lambda I_n - N \frac{I_m}{\lambda} N^T \right|$$

$$= \lambda^{m-n} \left| \lambda^2 I_n - N N^T \right| . \qquad (2.4)$$

Now

$$N N^T = \begin{bmatrix} 0 & -1 & -1 \ldots -1 & -1 \ldots -1 \\ -1 & 0 & -1 \ldots -1 & -1 \ldots -1 \\ -1 & -1 & 0 \ldots -1 & -1 \ldots -1 \\ . & & . & . \\ . & & . & . \\ . & & . & . \\ -1 & -1 & -1 \ldots \ 0 & -1 \ldots -1 \\ -1 & -1 & -1 \ldots -1 & -1 \ldots -1 \\ -1 & -1 & -1 \ldots -1 & -1 \ldots -1 \\ . & & . & . \\ . & & . & . \\ . & & . & . \\ -1 & -1 & -1 \ldots -1 & -1 \ldots -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \ldots -1 & -1 & -1 \ldots -1 \\ -1 & 0 & -1 \ldots -1 & -1 & -1 \ldots -1 \\ -1 & -1 & 0 \ldots -1 & -1 & -1 \ldots -1 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ -1 & -1 & -1 \ldots \ 0 & -1 & -1 \ldots -1 \\ -1 & -1 & -1 \ldots -1 & -1 & -1 \ldots -1 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ -1 & -1 & -1 \ldots -1 & -1 & -1 \ldots -1 \end{bmatrix}$$

$$
= \begin{bmatrix}
m-1 & m-2 & m-2 & \dots & m-2 & m-1 & m-1 & \dots & m-1 \\
m-2 & m-1 & m-2 & \dots & m-2 & m-1 & m-1 & \dots & m-1 \\
m-2 & m-2 & m-1 & \dots & m-2 & m-1 & m-1 & \dots & m-1 \\
\vdots & & & \vdots & & & & \vdots & \\
m-2 & m-2 & m-2 & \dots & m-1 & m-1 & m-1 & \dots & m-1 \\
m-1 & m-1 & m-1 & \dots & m-1 & m & m & \dots & m \\
m-1 & m-1 & m-1 & \dots & m-1 & m & m & \dots & m \\
\vdots & & & \vdots & & & & \vdots & \\
m-1 & m-1 & m-1 & \dots & m-1 & m & m & \dots & m
\end{bmatrix}.
$$

It is a square matrix of order n and it has a square submatrix of order (n – k) whose all elements are equal to m.

Substituting NN^T in (2.4), we get

$$
\lambda^{m-n} \begin{vmatrix}
\lambda^2+1-m & 2-m & 2-m & \dots & 2-m & 1-m & 1-m & \dots & 1-m \\
2-m & \lambda^2+1-m & 2-m & \dots & 2-m & 1-m & 1-m & \dots & 1-m \\
2-m & 2-m & \lambda^2+1-m & \dots & 2-m & 1-m & 1-m & \dots & 1-m \\
\vdots & & & \vdots & & & & \vdots & \\
2-m & 2-m & 2-m & \dots & \lambda^2+1-m & 1-m & 1-m & \dots & 1-m \\
1-m & 1-m & 1-m & \dots & 1-m & \lambda^2-m & -m & \dots & -m \\
1-m & 1-m & 1-m & \dots & 1-m & -m & \lambda^2-m & \dots & -m \\
\vdots & & & \vdots & & & & \vdots & \\
1-m & 1-m & 1-m & \dots & 1-m & -m & -m & \dots & \lambda^2-m
\end{vmatrix}. \qquad (2.5)
$$

Subtract column (k + 1) from the columns k + 2, k + 3, . . . , n of (2.5) to obtain (2.6).

$$
\lambda^{m-n} \begin{vmatrix}
\lambda^2+1-m & 2-m & 2-m & \dots & 2-m & 1-m & 0 & \dots & 0 \\
2-m & \lambda^2+1-m & 2-m & \dots & 2-m & 1-m & 0 & \dots & 0 \\
2-m & 2-m & \lambda^2+1-m & \dots & 2-m & 1-m & 0 & \dots & 0 \\
\vdots & & & \vdots & & & & \vdots & \\
2-m & 2-m & 2-m & \dots & \lambda^2+1-m & 1-m & 0 & \dots & 0 \\
1-m & 1-m & 1-m & \dots & 1-m & \lambda^2-m & -\lambda^2 & \dots & -\lambda^2 \\
1-m & 1-m & 1-m & \dots & 1-m & -m & \lambda^2 & \dots & 0 \\
\vdots & & & \vdots & & & & \vdots & \\
1-m & 1-m & 1-m & \dots & 1-m & -m & 0 & \dots & -\lambda^2
\end{vmatrix}. \qquad (2.6)
$$

Add rows k + 2, k + 3, . . . , n of (2.6) to its (k + 1)th row to obtain (2.7).

$$\lambda^{m-n}\begin{vmatrix} \lambda^2+1-m & 2-m & 2-m & \dots & 2-m & 1-m & 0 & \dots & 0 \\ 2-m & \lambda^2+1-m & 2-m & \dots & 2-m & 1-m & 0 & \dots & 0 \\ 2-m & 2-m & \lambda^2+1-m & \dots & 2-m & 1-m & 0 & \dots & 0 \\ \vdots & & & \vdots & & & & \vdots & \\ 2-m & 2-m & 2-m & \dots & \lambda^2+1-m & 1-m & 0 & \dots & 0 \\ (n-k)(1-m) & (n-k)(1-m) & (n-k)(1-m) & \dots & (n-k)(1-m) & \lambda^2-mn+mk & 0 & \dots & 0 \\ 1-m & 1-m & 1-m & \dots & 1-m & -m & \lambda^2 & \dots & 0 \\ \vdots & & & \vdots & & & & \vdots & \\ 1-m & 1-m & 1-m & \dots & 1-m & -m & 0 & \dots & -\lambda^2 \end{vmatrix} \quad (2.7)$$

It reduces to (2.8) in which the determinant is of order k + 1.

$$\lambda^{m-n}(\lambda^2)^{n-k-1}\begin{vmatrix} \lambda^2+1-m & 2-m & 2-m & \dots & 2-m & 1-m \\ 2-m & \lambda^2+1-m & 2-m & \dots & 2-m & 1-m \\ 2-m & 2-m & \lambda^2+1-m & \dots & 2-m & 1-m \\ \vdots & & & \vdots & & \\ 2-m & 2-m & 2-m & \dots & \lambda^2+1-m & 1-m \\ (n-k)(1-m) & (n-k)(1-m) & (n-k)(1-m) & \dots & (n-k)(1-m) & \lambda^2-mn+mk \end{vmatrix} \quad (2.8)$$

Subtract the first column of (2.8) from all its other columns to obtain (2.9).

$$\lambda^{m+n-2k-2}\begin{vmatrix} \lambda^2+1-m & 1-\lambda^2 & 1-\lambda^2 & \dots & 1-\lambda^2 & -\lambda^2 \\ 2-m & \lambda^2-1 & 0 & \dots & 0 & -1 \\ 2-m & 0 & \lambda^2-1 & \dots & 0 & -1 \\ \vdots & & & \vdots & & \\ 2-m & 0 & 0 & \dots & \lambda^2-1 & -1 \\ (n-k)(1-m) & 0 & 0 & \dots & 0 & \lambda^2-n+k \end{vmatrix}. \quad (2.9)$$

Add rows 2, 3, . . . , k of (2.9) to the first row to obtain (2.10).

$$\lambda^{m+n-2k-2}\begin{vmatrix} \lambda^2-mk+2k-1 & 0 & 0 & \dots & 0 & 1-k-\lambda^2 \\ 2-m & \lambda^2-1 & 0 & \dots & 0 & -1 \\ 2-m & 0 & \lambda^2-1 & \dots & 0 & -1 \\ \vdots & & & \vdots & & \\ 2-m & 0 & 0 & \dots & \lambda^2-1 & -1 \\ (n-k)(1-m) & 0 & 0 & \dots & 0 & \lambda^2-n+k \end{vmatrix}. \quad (2.10)$$

Add $(\lambda^2 + k - 1)/(\lambda^2 - n + k)$ times the $(k + 1)^{th}$ row of (2.10) to its first row to get (2.11).

$$\lambda^{m+n-2k-2}\begin{vmatrix} \lambda^2 - mk+2k-1+\dfrac{(n-k)(1-m)(\lambda^2 + k - 1)}{\lambda^2 - n + k} & 0 & 0 & \dots & 0 & 0 \\ 2-m & \lambda^2-1 & 0 & \dots & 0 & -1 \\ 2-m & 0 & \lambda^2-1 & \dots & 0 & -1 \\ . & & & . & & \\ . & & & . & & \\ . & & & . & & \\ 2-m & 0 & 0 & \dots & \lambda^2-1 & -1 \\ (n-k)(1-m) & 0 & 0 & \dots & 0 & \lambda^2-n+k \end{vmatrix} \qquad (2.11)$$

Expression (2.11) reduces to (2.12) in which the determinant is of order k.

$$\lambda^{m+n-2k-2}\left[\lambda^2 - mk+2k-1+\frac{(n-k)(1-m)((\lambda^2+k-1)}{\lambda^2 - n + k}\right]\begin{vmatrix} \lambda^2-1 & 0 & \dots & 0 & -1 \\ 0 & \lambda^2-1 & \dots & 0 & -1 \\ . & & . & & \\ . & & . & & \\ 0 & 0 & \dots & \lambda^2-1 & -1 \\ 0 & 0 & \dots & 0 & \lambda^2-n+k \end{vmatrix} \qquad (2.12)$$

$$= \lambda^{m+n-2k-2}\left[\lambda^2 - mk+2k-1+\frac{(n-k)(1-m)(\lambda^2+k-1)}{\lambda^2 - n + k}\right](\lambda^2-1)^{k-1}(\lambda^2 - n + k)$$

$$= \lambda^{m+n-2k-2}(\lambda^2 - 1)^{k-1}[\lambda^4 - (mn - 2k + 1)\lambda^2 + (m - k)(n - k)].$$

That equation (2.1) holds also for k = 0 is verified by direct calculation.

This completes the proof. □

<u>THEOREM 3:</u>

For m, n ≥ 1 and $0 \le r \le m$, $0 \le s \le n$,

$$\phi(Kb_{m,n}(r,s) : \lambda) = \lambda^{m+n-4}[\lambda^4 - (mn - rs)\lambda^2 + rs(m - r)(n - s)]. \qquad (2.13)$$

PROOF: Proceeds in a manner analogous to that of Theorem 2. For completeness we prove this in brief.

Let the vertex set of $K_{m,n}$ be partitioned into two disjoint sets $\{u_1, u_2, \dots, u_m\}$ and $\{v_1, v_2, \dots, v_n\}$ such that no two vertices in either sets are adjacent to each other. Without loss of generality assume that the edges of $K_{r,s}$ join the vertices u_i and v_j, for all i = 1, 2, . . . , r and j = 1, 2, . . . , s. Then the characteristic polynomial of $Kb_{m,n}(r,s)$ is the determinant (2.14).

$$
\begin{array}{c|ccccccccccccccc|}
 & u_1 & & & u_r & u_{r+1} & & u_m & v_1 & & v_s & v_{s+1} & & & & v_n \\
u_1 & \lambda & 0 & \dots & 0 & 0 & 0\dots & 0 & 0 & 0 & \dots\ 0 & -1 & -1 & \dots & & -1 \\
 & 0 & \lambda & \dots & 0 & 0 & 0\dots & 0 & 0 & 0 & \dots\ 0 & -1 & -1 & \dots & & -1 \\
 & \vdots & \vdots & & & & \vdots & & & & \vdots & & & & & \vdots \\
u_r & 0 & 0 & \dots & \lambda & 0 & 0\dots & 0 & 0 & 0 & \dots\ 0 & -1 & -1 & \dots & & -1 \\
u_{r+1} & 0 & 0 & \dots & 0 & \lambda & 0\dots & 0 & -1 & -1 & \dots\ -1 & -1 & -1 & \dots & & -1 \\
 & 0 & 0 & \dots & 0 & 0 & \lambda\dots & 0 & -1 & -1 & \dots\ -1 & -1 & -1 & \dots & & -1 \\
 & \vdots & \vdots & & & & \vdots & & & & \vdots & & & & & \vdots \\
u_m & 0 & 0 & \dots & 0 & 0 & 0\dots & \lambda & -1 & -1 & \dots\ -1 & -1 & -1 & \dots & & -1 \\
v_1 & 0 & 0 & \dots & 0 & -1 & -1\dots & -1 & \lambda & 0 & \dots\ 0 & 0 & 0 & \dots & & 0 \\
 & 0 & 0 & \dots & 0 & -1 & -1\dots & -1 & 0 & \lambda & \dots\ 0 & 0 & 0 & \dots & & 0 \\
 & \vdots & \vdots & & & & \vdots & & & & \vdots & & & & & \vdots \\
v_s & 0 & 0 & \dots & 0 & -1 & -1\dots & -1 & 0 & 0 & \dots\ \lambda & 0 & 0 & \dots & & 0 \\
v_{s+1} & -1 & -1 & \dots & -1 & -1 & -1\dots & -1 & 0 & 0 & \dots\ 0 & \lambda & 0 & \dots & & 0 \\
 & -1 & -1 & \dots & -1 & -1 & -1\dots & -1 & 0 & 0 & \dots\ 0 & 0 & \lambda & \dots & & 0 \\
 & \vdots & \vdots & & & & \vdots & & & & \vdots & & & & & \vdots \\
v_n & -1 & -1 & \dots & -1 & -1 & -1\dots & -1 & 0 & 0 & \dots\ 0 & 0 & 0 & \dots & & \lambda
\end{array}. \qquad (2.14)
$$

$$
= \begin{vmatrix} \lambda I_m & R^T \\ R & \lambda I_n \end{vmatrix} \qquad (2.15)
$$

Where

$$
R = \begin{bmatrix}
0 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\
0 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\
\vdots & & \vdots & & & & & \vdots \\
0 & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\
-1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\
-1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\
\vdots & & \vdots & & & & & \vdots \\
-1 & -1 & \dots & -1 & -1 & -1 & \dots & -1
\end{bmatrix}
$$

is a matrix of order $n \times m$ which contains a submatrix of order $s \times r$ whose all elements are equa to zero.

Applying Lemma 1 [2] to the expression (2.15), we get

$$\lambda^{m} \left| \lambda I_n - R \frac{I_m}{\lambda} R^T \right|$$

$$= \lambda^{m-n} \left| \lambda^2 I_n - R R^T \right| . \qquad (2.16)$$

Now

$$RR^T = \begin{bmatrix} m\text{-}r & m\text{-}r & \dots & m\text{-}r & m\text{-}r & m\text{-}r & \dots & m\text{-}r \\ m\text{-}r & m\text{-}r & \dots & m\text{-}r & m\text{-}r & m\text{-}r & \dots & m\text{-}r \\ m\text{-}r & m\text{-}r & \dots & m\text{-}r & m\text{-}r & m\text{-}r & \dots & m\text{-}r \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ m\text{-}r & m\text{-}r & \dots & m\text{-}r & m\text{-}r & m\text{-}r & \dots & m\text{-}r \\ m\text{-}r & m\text{-}r & \dots & m\text{-}r & m & m & \dots & m \\ m\text{-}r & m\text{-}r & \dots & m\text{-}r & m & m & \dots & m \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ m\text{-}r & m\text{-}r & \dots & m\text{-}r & m & m & \dots & m \end{bmatrix} .$$

It is a square matrix of order n in which there is a square submatrix of order (n – s) whose all elements are equal to m.

Substituting RR^T in (2.16), we get

$$\lambda^{m-n} \begin{vmatrix} \lambda^2+r\text{-}m & r\text{-}m & \dots & r\text{-}m & r\text{-}m & r\text{-}m & \dots & r\text{-}m \\ r\text{-}m & \lambda^2+r\text{-}m & \dots & r\text{-}m & r\text{-}m & r\text{-}m & \dots & r\text{-}m \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ r\text{-}m & r\text{-}m & \dots & \lambda^2+r\text{-}m & r\text{-}m & r\text{-}m & \dots & r\text{-}m \\ r\text{-}m & r\text{-}m & \dots & r\text{-}m & \lambda^2\text{-}m & \text{-}m & & \text{-}m \\ r\text{-}m & r\text{-}m & \dots & r\text{-}m & \text{-}m & \lambda^2\text{-}m & \dots & \text{-}m \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ \cdot & & \cdot & & & & \cdot & \\ r\text{-}m & r\text{-}m & \dots & r\text{-}m & \text{-}m & \text{-}m & \dots & \lambda^2\text{-}m \end{vmatrix} . \qquad (2.17)$$

Performing following operations on (2.17) we get (2.18).

(i) Subtract the first column from all its other columns

(ii) Subtract the $(s + 1)^{th}$ column from columns s + 2, s + 3, . . . , n.

(iii) Add rows s + 2, s + 3, . . . , n to the $(s + 1)^{th}$ row.

$$\lambda^{m-n}\begin{vmatrix} \lambda^2+r-m & -\lambda^2 & \dots & -\lambda^2 & -\lambda^2 & 0 & \dots & 0 \\ r-m & \lambda^2 & \dots & 0 & 0 & 0 & \dots & 0 \\ . & & . & & & & . & \\ . & & . & & & & . & \\ . & & . & & & & . & \\ r-m & 0 & \dots & \lambda^2 & 0 & 0 & \dots & 0 \\ (r-m)(n-s) & 0 & \dots & 0 & \lambda^2-r(n-s) & 0 & \dots & 0 \\ r-m & 0 & \dots & 0 & -r & \lambda^2 & \dots & 0 \\ . & & . & & & & . & \\ . & & . & & & & . & \\ . & & . & & & & . & \\ r-m & 0 & \dots & 0 & -r & 0 & \dots & \lambda^2 \end{vmatrix}. \quad (2.18)$$

$$= \lambda^{m-n}(\lambda^2)^{n-s-1}\begin{vmatrix} \lambda^2+r-m & -\lambda^2 & \dots & -\lambda^2 & -\lambda^2 \\ r-m & \lambda^2 & \dots & 0 & 0 \\ . & & . & & \\ . & & . & & \\ . & & . & & \\ r-m & 0 & \dots & \lambda^2 & 0 \\ (r-m)(n-s) & 0 & \dots & 0 & \lambda^2-r(n-s) \end{vmatrix} \quad (2.19)$$

Again performing following operations on the determinant (2.19) we get (2.20).

(i) Add rows 2, 3, . . . , s to the first row.

(ii) Add $\lambda^2/(\lambda^2-r(n-s))$ times the last row to its first row.

$$\lambda^{m+n-2s-2}\begin{vmatrix} \lambda^2+s(r-m)+\dfrac{(r-m)(n-s)\lambda^2}{\lambda^2-r(n-s)} & 0 & \dots & 0 & 0 \\ r-m & \lambda^2 & \dots & 0 & 0 \\ . & . & & & \\ . & . & & & \\ . & . & & & \\ r-m & 0 & \dots & \lambda^2 & 0 \\ (r-m)(n-s) & 0 & \dots & 0 & \lambda^2-r(n-s) \end{vmatrix} \quad (2.20)$$

$$= \lambda^{m+n-4}[\lambda^4-(mn-rs)\lambda^2+rs(m-r)(n-s)].$$

The equation (2.13) holds also for r = 0, s = 0 is verified by direct calculation.

This completes the proof. □

SPECTRA AND ENERGY OF $Ka_{m,n}(k)$ AND $Kb_{m,n}(r,s)$

From Theorems 2 and 3, it is elementary to obtain the spectra and energy of $Ka_{m,n}(k)$ and $Kb_{m,n}(r,s)$.

COROLLARY 4:

For m, n ≥ 1 and 0 ≤ k ≤ min{m, n}, the spectrum of $Ka_{m,n}(k)$ consists of 0 (m + n – 2k –2 times), 1 (k – 1 times), -1 (k – 1 times),

$$\pm\sqrt{\frac{mn - 2k + 1 + \sqrt{(mn - 2k + 1)^2 - 4(m - k)(n - k)}}{2}} \quad \text{and}$$

$$\pm\sqrt{\frac{mn - 2k + 1 - \sqrt{(mn - 2k + 1)^2 - 4(m - k)(n - k)}}{2}}\ . \qquad \square$$

COROLLARY 5:

For m, n ≥ 1 and 0 ≤ r ≤ m, 0 ≤ s ≤ n, the spectrum of $Kb_{m,n}(r,s)$ consists of 0 (m + n – 4 times),

$$\pm\sqrt{\frac{mn - rs + \sqrt{(mn - rs)^2 - 4rs(m - r)(n - s)}}{2}} \quad \text{and}$$

$$\pm\sqrt{\frac{mn - rs - \sqrt{(mn - rs)^2 - 4rs(m - r)(n - s)}}{2}}\ . \qquad \square$$

COROLLARY 6:

For m, n ≥ 1 and 0 ≤ k ≤ min{m, n},

$$E(Ka_{m,n}(k)) = 2k - 2 + 2\sqrt{mn - 2k + 1 + 2\sqrt{(m - k)(n - k)}}.$$

PROOF: From Theorem 2, the eigenvalues of $Ka_{m,n}(k)$ are 0 (m + n – 2k – 2 times), 1 (k – 1 times), -1 (k – 1 times) and the four roots of the equation

$\lambda^4 - (mn - 2k + 1)\lambda^2 + (m - k)(n - k) = 0.$

Let these four roots be λ_1, λ_2, λ_3, λ_4. Since $Ka_{m,n}(k)$ is a bipartite graph, by spectral property of bipartite graphs, for every eigenvalue λ of a bipartite graph there is -λ as its another eigenvalue [1].

Assume that $\lambda_1 = -\lambda_4$, and $\lambda_2 = -\lambda_3$.

Now $$\sum_{1\le i < j\le 4} \lambda_i \lambda_j = -(mn - 2k + 1)$$

$$\therefore \quad \lambda_1^2 + \lambda_2^2 = mn - 2k + 1$$

and $$\prod_{i=1}^{4} \lambda_i = (m - k)(n - k)$$

$\therefore \quad \lambda_1^2 \lambda_2^2 = (m-k)(n-k)$

$\therefore \quad \lambda_1 \lambda_2 = \sqrt{(m-k)(n-k)}$ (Only positive square root, since λ_1 and λ_2 are positive)

$\therefore \quad (\lambda_1 + \lambda_2)^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2$

$= mn - 2k + 1 + 2\sqrt{(m-k)(n-k)}$

$\therefore \quad \lambda_1 + \lambda_2 = \sqrt{mn - 2k + 1 + 2\sqrt{(m-k)(n-k)}}$

$\therefore E(Ka_{m,n}(k)) = |0|(m + n - 2k - 2) + |1|(k-1) + |-1|(k-1) + |\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4|$

$= k-1+k-1+2(|\lambda_1| + |\lambda_2|)$ (since $\lambda_1 = -\lambda_4$ and $\lambda_2 = -\lambda_3$)

$= 2k - 2 + 2|\lambda_1+\lambda_2|$ (since $\lambda_1, \lambda_2 \geq 0$)

$= 2k - 2 + 2\sqrt{mn - 2k + 1 + 2\sqrt{(m-k)(n-k)}}$. □

COROLLARY 7:

For $m, n \geq 1$ and $0 \leq r \leq m$, $0 \leq s \leq n$,

$E(Kb_{m,n}(r,s)) = 2\sqrt{mn - rs + 2\sqrt{rs(m-r)(n-s)}}$. □

Corollary 7 can be proven in a manner analogous to that of Corollary 6.

REMARKS

4.1 If $k = 0$, then the equation (2.1) reduces to $\lambda^{m+n-2}(\lambda^2 - mn)$ which is the characteristic polynomial of the complete bipartite graph $K_{m,n}$ [2, p.72].

4.2 If $m = n = k$, then the equation (2.1) reduces to $(\lambda^2 - 1)^{n-1}[\lambda^2 - (n-1)^2]$, which is the characteristic polynomial of $Ka_{n,n}(n)$ [see Figure 1].

$Ka_{3,3}(3)$ $Ka_{4,4}(4)$

Figure 1

4.3 If $r = 0$ or $s = 0$ or both, then the equation (2.13) reduces to the characteristic polynomial of $K_{m,n}$.

4.4 If $r = m$ and $s = n$, then the equation (2.13) reduces to λ^{m+n}, the characteristic polynomial of the complement of K_{m+n}.

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