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ENTANGLEMENT OF FORMATION DYNAMICS OF A TWO-MODE SYSTEM IN A COMMON GAUSSIAN NOISY CHANNEL

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ABSTRACT. Entanglement of formation is a measure that has a clear physically motivated definition, namely, it provides a lower bound over all pure-state entanglement decompositions required to create a given state. We investigate the open system dynamics of entanglement of formation of a two-mode Gaussian state interacting with a common bosonic environment. In the framework of the theory based on completely positive dynamical semigroups, the open system dynamics leads to a master equation in Lindblad form, which preserves the positivity and the Gaussian form of an initially Gaussian state at all times. For an initially separable state we witness the generation of entanglement, and its preservation in the limit of asymptotic times.

Keywords: entanglement of formation, common environment, Gaussian states.

INTRODUCTION

Quantum information theory has obtained remarkable achievements in the last decades, and as the second quantum revolution occurs, it promises even more groundbreaking advances. Important proposals in quantum cryptography ADESSO *et al.* (2014), teleportation and quantum computing have been already demonstrated in numerous experiments and have now entered the phase of technological applications and explorations. However, even though the processing and transmission of quantum information is currently very successful, one of the most pressing matters regarding the practical implementation of quantum communication and computation, represented by the fact that any physical quantum system is affected by a surrounding environment, is still intensely studied JEKNIC-DUGIC *et al.* (2016), JEKNIC-DUGIC *et al.* (2024). This is due to the fact that any quantum system must be regarded as an open system, that is subjected to interaction with an uncontrollable environment, which has a non-negligible influence on it. The theory of open quantum systems plays a major role in the study and application of quantum information. Quantum dynamics of open systems cannot be described in terms of a unitary time evolution, as opposed to the case of closed system, but by using a suitable equation of motion for its density operator, called quantum master equation (BREUER and PETRUCCIONE, 2006).

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In the last years, in-depth research has been conducted on the dynamics of Gaussian continuous-variable states embedded in an environment ARSENIJEVIC *et al*. (2018), MIHAESCU and ISAR (2018), STOICA and ISAR (2023), based on the fact that these states are readily generated and transformed using beam splitters, homodyne measurements, and squeezers in the laboratory XIANG *et al.* (2011a, 2011b). However, attempting to exploit these states for quantum information and computation purposes faces the obvious difficulty that they are unavoidably altered by the interaction with the environment GOYAL and GHOSH (2010). Therefore, investigation of the effects of quantum decoherence on the evolution of Gaussian continuous-variables states and, consequently, on their quantum correlations in noisy channels, is of great importance PRAUZNER-BECHCICKI and JAKUB (2004), TANG *et al.* (2005).

In JEONG *et al.* (2000) the authors studied the evolution of the nonlocality for an initial two-mode squeezed vacuum (TMSV) state in a thermal environment and have shown that the larger the initial squeezing parameter, the faster the squeezed state loses its nonlocality. For a similar initial TMSV state, but interacting with an uncorrelated squeezed environment, in TANG *et al.* (2005) it was revealed that squeezing the environment does not increase the lifetime of the entanglement of the initial two-mode squeezed state. In BENATTI and FLOREANINI (2006) the authors investigated two non-interacting harmonic oscillators interacting with a common bath and showed that starting from an initially separated Gaussian state, entanglement can be created during the interaction with the noisy environment. Regarding the generation of entanglement, the problem was studied further in XIANG *et al.* (2008), where two initial states were considered: a single-mode squeezed state and a squeezed vacuum state, interacting with a squeezed thermal environment. It was found that not only this type of environment can generate entanglement when starting with an uncorrelated state, but it can even amplify the existing entanglement, when the squeezing parameters are suitable chosen.

In this paper we investigate the entanglement of formation in Gaussian two-mode squeezed vacuum states, and its dynamics during the interaction with a common squeezed thermal bath. Entanglement of formation is a measure designed to quantify entanglement in mixed states in analogy to Schmidt decomposition criterion for pure states, whereby a separable global pure state can be factorized into local pure states. Otherwise, if the reduced states with respect to a given bipartition are mixed, then the global pure state is entangled and the von Neumann entropy represents a well-defined measure of entanglement. Entanglement of formation measure arises from the extension of the above concept to globally mixed states, where one has to evaluate the lower bound on entanglement given by the infimum over all possible pure-state decompositions WOOTTERS (2001). Thus, entanglement of formation is a generalized measure based on a well-motivated physical definition, which can be readily extended to multipartite states. Previously it was studied in Gaussian states undergoing the open system dynamics given by independent Gaussian noisy channels DUMITRU and ISAR (2015).

This paper is organized as follows. After the introduction to the microscopic derivation of the Lindblad master equation, we present the master equation tailored to optical systems, such as two bosonic modes interacting with a common bosonic environment. Then the time differential equation is solved for an initially separable two-mode Gaussian state, and the entanglement of formation is evaluated, highlighting the role of this setup on the entanglement generation processes. Then we present the results and discussions, altogether with the concluding remarks.

OPEN QUANTUM SYSTEMS: MASTER EQUATION

The theory of open quantum systems arose out of the need to describe the diffusion and dissipation in optical systems, for instance, and it deals with various quantum-statistical methods intended to describe the underlying physical processes governing the evolution. The

master equation approach leads to an effective equation of motion for a subsystem that belongs to a composed system: the system under investigation and the environment. Let us consider a system evolving according to a total Hamiltonian H composed of the free Hamiltonian H_0 and the interaction Hamiltonian H_1:

$$H = H_0 + H_1. \tag{1}$$

This decomposition is rather arbitrary, depending on the physical situation, where one may choose to highlight the physical process which is more influential on the dynamics of the system, compared to other underlying processes. The Schrödinger equation for a state $|\psi(t)\rangle$ describes the time evolution of the system:

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle \tag{2}$$

where the solution subject to the initial time constraint $|\psi(0)\rangle = |\psi(t=0)\rangle$ is given by $|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle$. This is the Schrödinger picture representation, where the time dependence is entirely carried by the state vector. Yet, a change of reference frame can highlight the contribution of the interaction Hamiltonian to changes in the state, $|\tilde{\psi}(t)\rangle = e^{-\frac{i}{\hbar}H_I t}|\psi(0)\rangle$, or equivalently $|\tilde{\psi}(t)\rangle = e^{-\frac{i}{\hbar}H_0 t}|\psi(t)\rangle$. In terms of the density operator the von Neumann equation governs the composite system evolution:

$$\frac{d}{dt}\rho(t) = -\frac{\mathrm{i}}{\hbar}[H, \rho(t)], \tag{3}$$

whereas the relation to the interaction picture representation $\tilde{\rho}(t)$ is given by: $\rho(t) = U_0(t)\,\tilde{\rho}(t)\,U_0^{\dagger}(t)$, where $U_0(t) = \mathrm{e}^{-\mathrm{i}H_0 t/\hbar}$ denotes the unitary transformation due to H_0. Combining this with Eqs. (1) and (3) one obtains:

$$\begin{aligned}\frac{d}{dt}U_0(t)\,\tilde{\rho}(t)\,U_0^{\dagger}(t) &= -\frac{\mathrm{i}}{\hbar}[H_0, U_0(t)\,\tilde{\rho}(t)\,U_0^{\dagger}(t)] + U_0(t)\,\frac{d}{dt}\tilde{\rho}(t)\,U_0^{\dagger}(t) \\ &= -\frac{\mathrm{i}}{\hbar}[H, U_0(t)\,\tilde{\rho}(t)\,U_0^{\dagger}(t)],\end{aligned} \tag{4}$$

arriving at the von Neumann equation in the interaction picture:

$$\frac{d}{dt}\tilde{\rho}(t) = -\frac{\mathrm{i}}{\hbar}[H_I, \tilde{\rho}(t)], \tag{5}$$

where H_I denotes the interaction Hamiltonian, defined as:

$$H_I = U_0^{\dagger}(t)\,H_1\,U_0(t), \tag{6}$$

and we used the relations $[H_0, U_0(t)] = 0$ and $U_0^{-1}(t) = U_0^{\dagger}(t)$. A formal solution to Eq. (5) is given by

$$\tilde{\rho}(t) = \rho(0) + \frac{1}{\mathrm{i}\hbar}\int_0^t dt'\,[H_I(t'), \tilde{\rho}(t')], \tag{7}$$

which now can be substituted into Eq. (3), arriving at the von Neumann equation in the integro-differential form in the interaction picture:

$$\frac{d}{dt}\tilde{\rho}(t) = \frac{1}{i\hbar}[H_I(t), \rho(0)] - \frac{1}{\hbar^2}\int_0^t dt' \, [H_I(t), [H_I(t'), \tilde{\rho}(t')]]. \tag{8}$$

Born and Markov approximations

From Eq. (8) we see that the evolution of the composite system is governed by the exact master equation, determined by the Hamiltonian that couples the system to the environment. However, one is interested in the information about the system, while the reservoir is known usually only through its general characteristics, such as temperature and energy density of states. One recovers the evolution of the system of interest through tracing out the degrees of freedom of the environment. By denoting $\tilde{\rho}_S(t) = \mathrm{Tr}_R[\tilde{\rho}(t)]$, we obtain

$$\frac{d}{dt}\tilde{\rho}_S(t) = \frac{1}{i\hbar}\mathrm{Tr}_R\{[H_I(t), \rho(0)]\} - \frac{1}{\hbar^2}\int_0^t dt' \, \mathrm{Tr}_R\{[H_I(t), [H_I(t'), \tilde{\rho}(t')]]\}, \tag{9}$$

where the subscripts S and R stand for the system and reservoir, respectively.

In addition, the following assumptions are required in order to achieve the well-known Lindblad master equation.

1. ***Born approximation I.*** We consider that at the initial time $t_0 = 0$ the system and the environment are uncorrelated:

   $$\rho(0) = \rho_S(0) \otimes \rho_R(0). \tag{10}$$

   Additionally, one can assign $\mathrm{Tr}_R\{[H_I(t), \rho_R(0)]\} = 0$ if the reservoir operators coupled to the system have zero mean value in the state $\rho_R(0)$, which can be adjusted by including these average values in the system Hamiltonian.
2. ***Born approximation II.*** Inevitably the system and environment become correlated in time, yet this effect can be neglected if one considers a weak coupling between them, along with the assumption that the environment is much larger than the system. In this case, the reservoir initially considered in thermal equilibrium state, stays unperturbed by the interaction with the system, so that the factorization in Eq. (9) applies also at later times: $\rho(t) = \rho_S(t) \otimes \rho_R(0)$.
3. ***Markovian approximation.*** Finally, in the axiomatic approach, the most general generator of the Markovian dynamics is obtained on the base of completely positive quantum dynamical semigroups, assuring that the memory effects from the previous times are neglected. Similarly, in Eq. (9), where the time dependent state of the system depends on its past history through the integration of $\rho_S(t')$, it is replaced by $\rho_S(t)$. In addition, the time evolution of the reduced density operator still depends upon an explicit choice for the initial preparation at time $t = 0$, and, therefore, this dynamics is not yet represented by a dynamical semigroup. Therefore, a change of integration variable and of the limits of integration, respectively

   $$t' \to t - t', \qquad \int_0^t \to \int_0^\infty, \tag{11}$$

will restore the equivalence with the Lindblad master equation obtained in the framework of completely positive quantum dynamical semigroups.

Finally, the Lindblad master equation in the interaction picture reads:

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\frac{1}{\hbar^2}\int_0^\infty dt'\ \mathrm{Tr}_R\{[H_I(t),[H_I(t-t'),\tilde{\rho}_S(t)\otimes\rho_R(0)]]\}. \tag{12}$$

Nowadays the Lindblad master equation is a foundational tool for the open quantum system dynamics modeling, in various research areas. Detailed discussions on the validity boundaries of Eq. (12) are offered in Refs. CARMICHAEL (1999) and (BREUER and PETRUCCIONE, 2006).

OPTICAL MASTER EQUATION: DAMPED HARMONIC OSCILLATORS

The two basic building block systems in quantum optics are the damped harmonic oscillator and the damped two-level atom. The investigation here covers two bosonic modes interacting with a common environment, which is treated as a reservoir composed of an infinite number of bosonic modes. Commonly, the infinite collection of environmental bosonic modes is considered to be in thermal equilibrium state, however a more general environment is obtained once we consider the phase sensitive states, i.e. squeezed thermal states. The free Hamiltonian in this case gives the free evolution of the system and the thermal bath

$$\begin{aligned} &H_0 = H_S + H_B, \\ &H_S = \hbar\sum_{i=1}^{2}\omega_i a_i^\dagger a_i, \qquad\qquad H_B = \hbar\sum_k \Omega_k b_k^\dagger b_k, \end{aligned} \tag{13}$$

where a_i and b_k denote, respectively, the annihilation operators of the two bosonic modes of the system with frequencies ω_i and the reservoir with mode frequencies Ω_k, while the sum over k accounts for the infinite number of bosonic modes in the environment. For defining the interaction Hamiltonian, we have to include the terms that describe two bosonic modes interacting with a common environment. This setup emerges from the following interaction Hamiltonian:

$$H_1 = \sum\nolimits_k g_k b_k (a_1^\dagger + a_2^\dagger) + h.c. \tag{14}$$

It is important to note that we assumed the Rotating Wave Approximation (RWA) in the interaction Hamiltonian, which essentially neglects the fastly oscillating terms. The contribution of these terms to the dynamics of the system is negligible, since they perform in different time scales CARMICHAEL (1999).

Using the Hamiltonian (14) in the master equation (12), we arrive at the Lindblad master equation in interaction picture ($\hbar = 1$) XIANG *et al.* (2011a):

$$\begin{aligned} \frac{d}{dt}\rho_S = \frac{\gamma}{2}\sum_{i,j=1,2}\{&(N+1)\left(2a_i\rho a_j^+ - a_j^+ a_i\rho - \rho a_j^+ a_i\right) + N\left(2a_j^+\rho a_i - a_i a_j^+\rho - \rho a_i a_j^+\right) \\ &+ M\left(2a_i^+\rho a_j^+ - a_j^+ a_i^+\rho - \rho a_j^+ a_i^+\right) + M^*\left(2a_i\rho a_j - a_j a_i\rho - \rho a_j a_i\right)\}, \end{aligned} \tag{15}$$

where γ is the dissipation parameter and the coefficients N and M are related to the correlation functions of the environment, once the time integral and partial trace on the environment

degrees of freedom are performed in Eq. (12), and the resonant terms survive SCULLY *et al.* (1997):

$$\begin{aligned}
&\left\langle b^{+}(\Omega_i)\, b(\Omega_j) \right\rangle = N(\Omega_i)\, \delta(\Omega_i - \Omega_j)\,, \\
&\left\langle b(\Omega_i)\, b^{+}(\Omega_j) \right\rangle = \left(N(\Omega_i) + 1 \right) \delta(\Omega_i - \Omega_j)\,, \\
&\left\langle b(\Omega_i)\, b(\Omega_j) \right\rangle = M(\Omega_i)\, \delta(\Omega_i - \Omega_j)\,.
\end{aligned} \tag{16}$$

These parameters satisfy the relation $|M|^2 \leq N(N+1)$, which assures the complete positivity condition of the Lindblad master equation above. In particular, for a squeezed vacuum bath $|M|^2 = N(N+1)$, and the coefficients are given by:

$$N = \sinh^2 s, \qquad M = -e^{i\theta} \cosh s \sinh s, \tag{17}$$

where s is the squeezing parameter of the reservoir, and θ is the squeezing angle. From here on we consider $\theta = 0$, and therefore, M is a real number.

Gaussian states

The system of two uncoupled harmonic oscillators pertains to the class of continuous variable (CV) states described in an infinite dimensional Hilbert space by pairs of ladder operators for each mode or, alternatively, in terms of position and momentum operators, related as $a_k = (x_k + p_k)/\sqrt{2}$. Arranged in a vector of operators $R^{\mathrm{T}} = (x_1, p_1, x_2, p_2)$, the canonical commutation relation has the following form, where J_1 is the one mode symplectic matrix:

$$[R_k, R_l] = \mathrm{i}\, J_{kl}, \qquad J = J_1 \oplus J_1, \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{18}$$

The CV states commonly encountered in physical setups are Gaussian states defined as the eigenstates of a quadratic Hamiltonian, an example of which is the Hamiltonian of harmonic oscillators. In phase space Gaussian states are fully characterized by a characteristic function of Gaussian form SERAFINI *et al.* (2004):

$$\chi_G(\xi) = e^{-\frac{1}{2}\xi^{\mathrm{T}}\sigma\xi - \mathrm{i}\xi^{\mathrm{T}}d}, \tag{19}$$

which is completely described by the first and second statistical moments, namely, the displacement vector d and the covariance matrix σ, respectively, given by:

$$d = \mathrm{Tr}[\rho_G R], \qquad \sigma = \mathrm{Tr}[\{(R-d)\,, (R-d)^{\mathrm{T}}\}_{+} \rho_G], \tag{20}$$

where ρ_G is the density operator of the Gaussian state. While the displacements can be adjusted to zero, the covariance matrix encodes all the information on quantum correlations, such as entanglement. The block decomposition of a bipartite covariance matrix is defined as:

$$\sigma = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \tag{21}$$

where A and B are the covariance matrices of the first and second mode, respectively, and C is the correlation matrix between the bosonic modes.

The initial Gaussian state evolving in time under the Lindblad master equation will remain Gaussian at all times, since Eq. (15) involves only terms which are utmost quadratic

PARIS *et al.* (2003). Typically, in this case one has to rewrite the master equation for the characteristic function through the standard operator correspondence, leading to the so-called Lyapunov equation for the covariance matrix.

Initial single-mode squeezed state

The two-mode covariance matrix of the single-mode squeezed state is given by $\sigma_{\mathrm{SMSS}}=\sigma_s\oplus\sigma_s$, where σ_s is the single-mode covariance matrix:

$$\sigma_s=\frac{1}{2}\begin{pmatrix}\cosh 2r & \sinh 2r\\ \sinh 2r & \cosh 2r\end{pmatrix}, \tag{22}$$

where r is the squeezing parameter of the state. The time-depending solution for the covariance in this case can be brought to the form XIANG *et al.* (2008), XIANG *et al.* (2009):

$$\sigma^{I}_{\mathrm{SMSS}}(t)=\frac{1}{2}\begin{pmatrix}n_1 & 0 & c_1 & 0\\ 0 & n_2 & 0 & c_2\\ c_1 & 0 & n_1 & 0\\ 0 & c_2 & 0 & n_2\end{pmatrix}, \tag{23}$$

with the following matrix elements:

$$\begin{aligned} n_1&=\frac{1}{2}\left[\left(2N+1-e^{2r}+2M\right)\tau+2e^{2r}\right], \quad n_2=\frac{1}{2}\left[\left(2N+1-e^{-2r}-2M\right)\tau+2e^{-2r}\right],\\ c_1&=\frac{1}{2}\left[2N+1+2M-e^{2r}\right]\tau, \qquad\qquad c_2=\frac{1}{2}\left[2N+1-2M-e^{-2r}\right]\tau, \end{aligned} \tag{24}$$

where the time parametrization is encoded in the dimensionless time variable $\tau=1-e^{-2\gamma t}$.

ENTANGLEMENT OF FORMATION

Entanglement of formation is a measure of entanglement that is based on an operational definition, namely, the quantification of how much pure state entanglement is required to create a given state. Its physical motivation relies on defining a reliable mixed state entanglement quantifier based on the fact that any given state ρ can be decomposed in the basis of pure maximally entangled states as

$$\rho=\Sigma_j p_j\,|\psi_j\rangle\langle\psi_j|, \quad \text{with} \quad \sum_j p_j=1. \tag{25}$$

Then, entanglement of formation is given by the infimum pure state decompositions, i.e as a weighted sum over the von Neumann entropy $E(\Psi_j)$ of the maximally entangled pure state Ψ_j WOOTTERS (2001):

$$E_f(\rho)=\inf\sum_j p_j E(\Psi_j)\,. \tag{26}$$

Calculating all such possible decompositions is a daunting task, however, for the class of symmetric Gaussian states, which do not change after interchanging the two modes, a generic formula has been deduced GIEDKE *et al.* (2003):

$$E_f = h_+(\Delta)\log_2[h_+(\Delta)] - h_-(\Delta)\log_2[h_-(\Delta)], \tag{27}$$

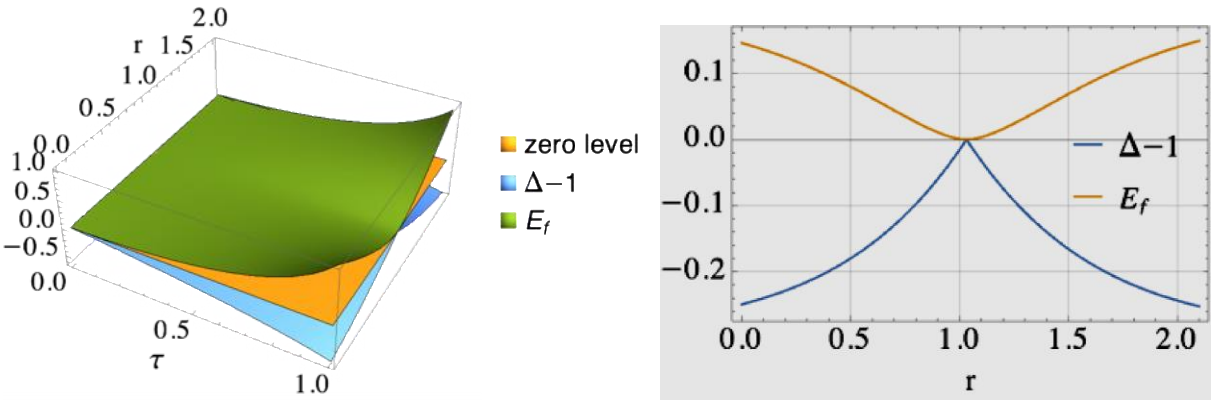
where $h_\pm(\Delta) = (\Delta^{-1/2} \pm \Delta^{1/2})^2/4$, and the function Δ can be expressed in terms of the block matrices of the covariance matrix (21), as follows RIGOLIN *et al.* (2004):

$$\Delta = 2\sqrt{\det A - \det C - \sqrt{\mathrm{Tr}(AJCJBJC^TJ) - 2\det A \det C}}\,. \tag{28}$$

This measure of entanglement represents a positive definite function and a reliable quantifier however, it must be checked together with the actual witness of entanglement, which is the function Δ. Thus, a given state is entangled if and only if $\Delta < 1$, while in the case of $\Delta \geq 1$ the value of E_f is no longer representative for the presence of quantum entanglement.

RESULTS AND DISCUSSIONS

In Figs. 1-3 it is shown the function $\Delta - 1$ together with the entanglement of formation measure E_f for the initial bipartite single mode squeezed state, which is a separable state. As discussed in the previous subsection, the presence of entanglement is certified whenever $\Delta - 1 < 0$, and in this case E_f quantifies the amount of entanglement. Thus, a notable result of these assertions is that the initially separable bipartite state can acquire entanglement due to the interaction with a common squeezed vacuum environment or squeezed thermal environment, depending on the interplay between bath parameters, such as squeezing and temperature, and squeezing of the initial state.



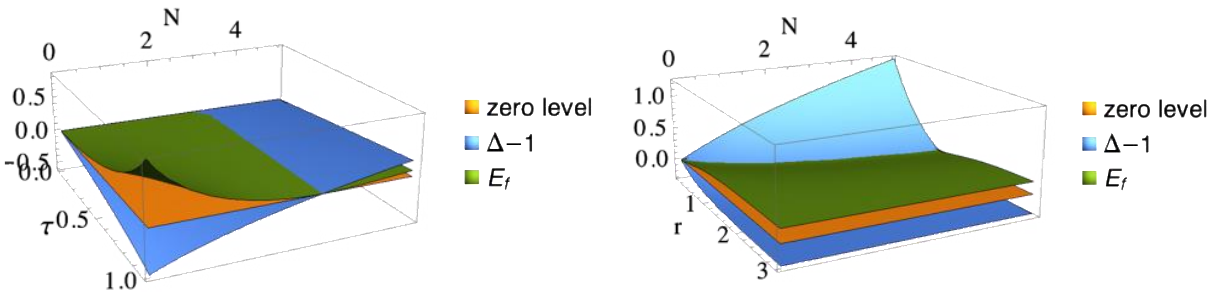
(a) (b)

Figure 1. Dependence of the entanglement of formation and function $\Delta - 1$ on the squeezing parameter r for an initial single-mode squeezed state in a squeezed vacuum bath with $N = 1.5$. In (a) it is shown also the time dependence, where $\tau = 1 - e^{-2\gamma t}$ is the dimensionless time parameter, and in (b) the instant of time is fixed to $\tau = 0.5$.

Fig. 1 (a) depicts the time evolution of entanglement of formation, and its dependence on the initial state squeezing for a squeezed vacuum environment (temperature $T = 0$), while Fig. 1 (b) captures an instance of dimensionless time for the same parameters as in Fig. 1 (a). The central point in the plot, where $E_f = \Delta - 1 = 0$ always, corresponds to equal squeezing parameters s of the bath and r of the system and, therefore, these two opposed contributions counter compensate each other. Entanglement is generated, it increases monotonically in time and evolves non-monotonically around this equal squeezing point, where $\Delta - 1 < 0$ and the difference between the two mentioned squeezing parameters determines the values of the

created. One can see that E_f is a positive function achieving its maximum in the limit of asymptotic time ($\tau = 1$).

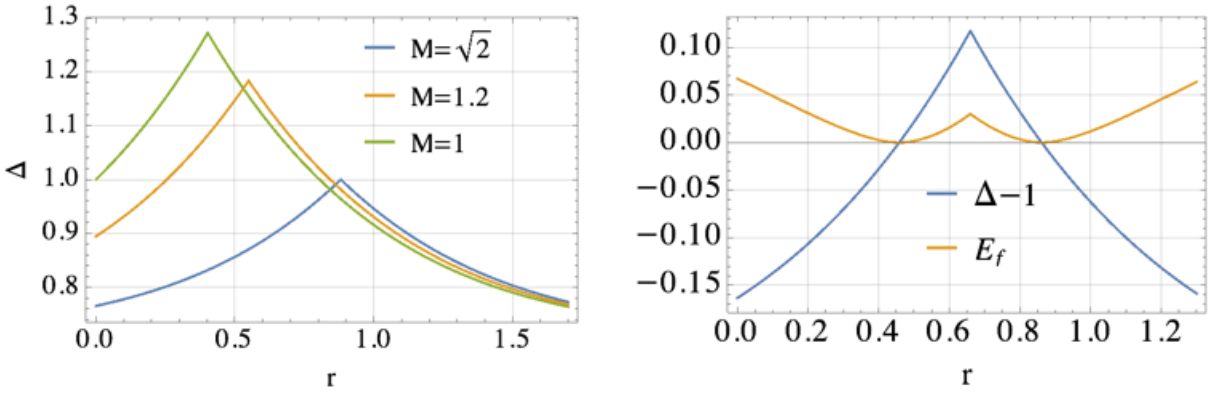
Figs. 2 (a) and (b) depict the time evolution of entanglement for a thermal bath as a function of the thermal photon number of the environment, which in this case depends only on the temperature of the environment, since the bath squeezing is zero. Similarly, as in the previous pictures, one can see that entanglement generation takes place, due to the effective communication established between the two modes, where the environment plays the role of a mediator. This mechanism operates on the resources of the initial state, as the single mode squeezing in our case. However, the creation of entanglement is impeded, for relatively large temperatures, by the quantum decoherence due to interaction with the bath. One can see in Fig. 2 (b) that higher the initial state squeezing mitigates the destruction effect of the high thermal photon number in the reservoir and leads to the preservation of entanglement.



(a) (b)

Figure 2. Dependence of the entanglement of formation and function $\Delta - 1$ on the thermal photon number N for an initial single-mode squeezed state interacting with a thermal bath ($M = 0$).
(a) Time dependence for a fixed squeezing parameter $r = 1$, and
(b) dependence on the initial squeezing at a fixed instance of time $\tau = 0.5$, where $\tau = 1 - e^{-2\gamma t}$ is the dimensionless time parameter.

Fig. 3 (a) shows the dependence of the function Δ and Fig. 3 (b) shows the dependence of the function $\Delta - 1$ and of the entanglement of formation on the initial state squeezing for different temperatures of the squeezed thermal reservoir.



(a) (b)

Figure 3. Dependence on squeezing parameter r of the initial single-mode squeezed state in a squeezed thermal bath with $N = 1$, at the instance of dimensionless time $\tau = 0.5$, (a) of the function Δ for different values of bath squeezing M, and (b) of the entanglement of formation and function $\Delta - 1$ for $M = 1.3$.

In this case $|M|^2 < N(N+1)$, and higher is the gap in this inequality, higher is the temperature of the environment, which can be detrimental to the creation and preservation of entanglement. Thus, compared to the squeezed vacuum bath (Fig. 1 and the blue line in Fig. 3 (a)), a non-zero value of the temperature of the bath leads to the appearance of a gap in the region where $\Delta < 1$ and where entanglement is created, around the point where the squeezing parameters of the initial state and of the bath are equal, corresponding to the peak on the blue line in Fig. 3 (a).

This region becomes larger by increasing the temperature, so that for temperatures high enough, the entanglement can be generated only for stronger initial squeezing parameter than some critical value, which is close to the equal squeezing value. In Figs. 3 (a) and (b) we can observe that even though entanglement of formation measure has positive values, it is not representative of entanglement if $\Delta - 1 > 0$.

CONCLUSIONS

We investigated the open system dynamics of a bipartite Gaussian state interacting with a common squeezed thermal bath, focusing on the time evolution of quantum entanglement quantified by the entanglement of formation measure. We briefly outlined the microscopic derivation method leading to the Markovian master equation in Lindblad form, and presented the optical master equation corresponding to two bosonic modes interacting with a common squeezed bosonic environment.

Compared to the previously studied evolution of entanglement of formation in Gaussian states coupled to two independent environments, the common reservoir, being a particular case of two correlated baths, plays the role of a communication mediator between the two modes of the system, which amounts to generation of entanglement in an initially separable state. Yet, the interaction with the environment tends to impede the entanglement creation and preservation. This was revealed by the opposing actions provided by the squeezing of the initial state and the squeezing parameter of the environment, as well as by the influence of the non-zero temperature of the bath. In Ref. XIANG *et al.* (2008) there were presented similar results for the behaviour of the logarithmic negativity measure of entanglement in the common reservoir setup. This similarity is not surprising, given that both measures, logarithmic negativity and entanglement of formation, are viable quantifiers of entanglement. Compared to logarithmic negativity, entanglement of formation has an operational definition with a clear physically motivated derivation, which could be straightforwardly investigated for multi-mode states, and therefore, we appreciate that it can present an interesting and important subject of future studies.

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