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ABOUT THERMODYNAMICS OF SCHWARZSCHILD BLACK HOLES IN A THREE-DIMENSIONAL QUANTUM VACUUM WITH GENERALIZED UNCERTAINTY RELATIONS

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ABSTRACT. By considering a modified version of generalized uncertainty relations in the model of a three-dimensional dynamic quantum vacuum characterized by a variable energy density, new relevant perspectives of analysis of the thermodynamics of Schwarzschild black holes are explored. The dependence of temperature and entropy of a Schwarzschild black hole with the variable quantum vacuum energy density is analyzed. Finally, the thermodynamics of Schwarzschild black holes surrounded by quintessence is studied in a picture where quintessence, and thus dark energy, is the manifestation of more elementary quantum vacuum energy density fluctuations as well as a state parameter of the vacuum.

Keywords: generalized uncertainty relations, three-dimensional dynamic quantum vacuum, variable quantum vacuum energy density, Schwarzschild black holes.

INTRODUCTION

One of the most relevant and appealing consequences of quantum gravity theories lies in the existence of a minimal measurable length that naturally generates a modification of the Heisenberg position-momentum uncertainty relations at the Planck scale. Different proposals of modification of the Heisenberg uncertainty relations exist which introduce a minimal-length scenario (MAGGIORE, 1993; KEMPF *et al.*, 1995; ADLER and SANTIAGO, 1999; SCARDIGLI, 1999; CAPOZZIELLO *et al.*, 2000; CARR, 2015). These modified versions of the uncertainty principle, which are known today with the expression Generalized Uncertainty Principle (GUP), introduce important perspectives in various areas of theoretical physics. In particular, they reveal themselves as an adequate instrument in order to study the thermodynamic properties of black holes.

The influence of GUP at the Planck scale in the changes of the thermodynamic properties of a black hole has been analysed in different contexts, such as in reference to the formation of mini black holes in a consistent way with doubly special relativity (ALI, 2012;

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PRAMANIK *et al.,* 2015), or in the exploration of the properties of quantum black holes in the picture of a deformed version of Wheeler-DeWitt equation (BINA *et al.*, 2010; MAJUMDER, 2011). The impact of GUP in the dynamics of the universe manifests itself also in determining a correction to the relation between entropy and area, which turns out to be universal for all black objects (FAIZAL and KHALIL, 2015) and in the context of black hole complementarity (CHEN *et al.*, 2014). Furthermore, it has been recently shown that a peculiar high-order generalized uncertainty principle produces a modification of temperature, entropy, and capacity of a semi-classical black hole (HASSANABADI *et al.*, 2019a).

In this paper, the main purpose is to investigate how, in the model of a three-dimensional (3D) dynamic quantum vacuum (DQV) proposed recently by the author, a special form of extended generalized uncertainty relations can be formulated which leads to a new re-reading of the thermodynamics of Schwarzschild black holes. This work is structured in the following way. In section 2, after a review of the fundamental features of the model of the 3D DQV, we enunciate our extended generalized uncertainty relations. In section 3, after analysing the dependence of the mass of a Schwarzschild black hole with the variable quantum vacuum energy density, we explore how the thermodynamic quantities of a Schwarzschild black hole, namely temperature and entropy, are modified by the extended generalized uncertainty relations in the 3D DQV model and we discuss their physical meaning. In section 4 we explore the consequences of the generalized uncertainty relations of the 3D DQV in the treatment of the thermodynamics of Schwarzschild black holes in the presence of quintessence. In section 5 we summarize the main results of the paper and we analyse the relation of our model with other relevant alternative approaches which invoke the existence of minimal lengths and with current models of thermodynamics of black holes surrounded by quintessence matter.

THE GENERALIZED UNCERTAINTY RELATIONS IN THE THREE-DIMENSIONAL DYNAMIC QUANTUM VACUUM

By considering, for mirror-symmetric states, the deformed canonical commutation relations expressed by

$$\left[\hat{X},\hat{P}\right] = i\hbar\left(1+\beta l_p^2\frac{\hat{P}^2}{\hbar^2}\right) \tag{1}$$

where $\hat{P}^2 = \sum_k \hat{P}_k^2$ and $\hat{X},\hat{P}$ represent the high-energy position and momentum operators, in PETRUZZIELLO and ILLUMINATI (2021) a generalized uncertainty relation has been proposed, which predicts the existence of a minimal length, of the form

$$\Delta x\Delta p \geq \frac{\hbar}{2}\left[1+\beta l_p^2\left(\frac{\Delta p}{\hbar}\right)^2\right] \tag{2}$$

where β is a deformation parameter expressing the space-time fluctuations at the Planck scale and l_p is the Planck length. The generalized uncertainty relation (2) has an important impact as regards the explanation of relevant aspects of quantum measurement in a picture where the decoherence rate turns out to be minimal in the deep quantum regime below the Planck scale and maximal further than the mesoscopic regime. Moreover, it leads to the recover of the standard quantum mechanical dynamics of a more general approach in the limit $\beta \rightarrow 0$.

In the model of the 3D DQV proposed by the author in several recent works (FISCALETTI and SORLI, 2014a, 2014b, 2016a, 2016b, 2016c, 2017, 2018; FISCALETTI, 2015, 2016a, 2016b, 2020) one has the possibility to introduce a new form of generalized uncertainty relations where the quantity $\left[1+\beta l_p^2\left(\frac{\Delta p}{\hbar}\right)^2\right]$ appearing in equation (2) can get a new interpretation, at a deeper level, in terms of the fundamental variable energy density which characterizes the 3D DQV. Our model of 3D DQV has the merit to derive ordinary matter, dark matter and dark energy as special states of the 3D quantum vacuum, defined by a Planckian metric and a variable quantum vacuum energy density, where time does not exist as a primary physical reality but emerges as a mathematical parameter measuring only the sequential numerical order of material changes and, in this picture, the variable energy density can be associated to a deformation of the geometry of the underlying background which, at a fundamental level, is expressed by generalized uncertainty relations.

The fundamental 3D DQV is defined by a ground state which is characterized by the maximum value of the quantum vacuum energy density given by the Planck energy density

$$\rho_{pE} = \frac{M_{Pl}c^2}{l_p^3} = 4{,}641266 \cdot 10^{113} J/m^3 \quad (3)$$

(where M_{Pl} is Planck's mass, c is the light speed and l_p is Planck's length). The appearance of ordinary baryonic matter derives from an opportune excited state of the 3D quantum vacuum which is characterized by an opportune diminishing of the quantum vacuum energy density associated to elementary reduction-state (RS) processes of creation/annihilation of virtual particle/antiparticle pairs, given by relation

$$\Delta\rho_{qvE} \equiv \rho_{pE} - \rho = \frac{mc^2}{V} \quad (4)$$

with respect to the ground state, depending on the amount of mass m and the volume V of the particle. Each excited state of the DQV can be described by a wave function $C = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$ at two components satisfying a time-symmetric extension of the Klein-Gordon quantum relativistic equation

$$\begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix} C = 0 \quad (5)$$

where $H = \left(-\hbar^2 \partial^\mu \partial_\mu + \frac{V^2}{c^2} \left(\Delta\rho_{qvE}\right)^2 \right)$ and $\Delta\rho_{qvE}$ is the change of the quantum vacuum energy density, provides a mathematical description of each excited state of the DQV in a picture where the quantum potential of the vacuum

$$Q_{Q,i} = \frac{\hbar^2 c^2}{V^2 \left(\Delta\rho_{qvE}\right)^2} \left(\frac{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) \left|\psi_{Q,i}\right|}{\left|\psi_{Q,i}\right|} - \frac{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) \left|\phi_{Q,i}\right|}{\left|\phi_{Q,i}\right|} \right) \quad (6)$$

emerges as the ultimate entity guiding the occurrence of the processes of creation or annihilation events in space and gives origin to a fundamental non-local character of the background (FISCALETTI and SORLI, 2014b, 2017; FISCALETTI, 2016b).

Dark energy cannot be considered as a primary physical reality, but its action is an emerging process that is generated by opportune quantum vacuum energy density fluctuations $\Delta\rho_{qvE}^{DE}$ on the basis of relation

$$\rho_{DE} \cong \frac{35Gc^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6 \tag{7}$$

The dark energy density (7), which can be associated to the action of a cosmological constant, is responsible of the curvature of space-time in the sense that the geometry of the 3D quantum vacuum can be described by an underlying quantized metric

$$d\hat{s}^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

where

$$\hat{g}_{00} = -1 + \hat{h}_{00},\ \hat{g}_{11} = 1 + \hat{h}_{11},\ \hat{g}_{22} = r^2\left(1 + \hat{h}_{22}\right),\ \hat{g}_{33} = r^2 \sin^2\vartheta\left(1 + \hat{h}_{33}\right),\ \hat{g}_{\mu\nu} = \hat{h}_{\mu\nu} \text{ for } \mu \neq \nu \tag{9}$$

and, at the order $O\left(r^2\right)$,

$$\left\langle \hat{h}_{\mu\nu} \right\rangle = 0 \text{ except } \left\langle \hat{h}_{00} \right\rangle = \frac{8\pi G}{3}\left(\frac{\Delta\rho_{qvE}}{c^2} + \frac{35Gc^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6\right)r^2 \text{ and}$$

$$\left\langle \hat{h}_{11} \right\rangle = \frac{8\pi G}{3}\left(-\frac{\Delta\rho_{qvE}}{2c^2} + \frac{35Gc^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6\right)r^2 \tag{10}$$

(FISCALETTI, 2016a, 2016b, 2020; FISCALETTI and SORLI, 2016a, 2016b, 2016c).

Now, in our model, we assume that the term $\left[1 + \beta l_p^2 \left(\frac{\Delta p}{\hbar}\right)^2\right]$ appearing in the formalism (2) of the generalized uncertainty relations finds its origin in more fundamental properties, namely the fluctuations of the quantum vacuum energy density (4). As a consequence, the quantity $\left(\frac{\Delta p}{\hbar}\right)^2$ in our model becomes $\frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}$ and therefore, in the 3D DQV, one can introduce extended generalized uncertainty relations, which are valid at the Planck scale, of the form

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1 + \beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}\right) \tag{11}$$

In equation (11) the parameter β is a fluctuating quantity which is associated with space-time fluctuations at the Planck scale, in affinity with the treatment developed in quantum foam scenarios such as loop quantum gravity and cellular automaton interpretation of quantum mechanics (AMELINO-CAMELIA, 2002; ROVELLI, 2010; FISCALETTI, 2014; 'T HOOFT, 2001a, 2001b; 2013, 2016; LICATA, 2020).

The generalized uncertainty relations (11) have the merit to introduce new scenarios of connection between great theories of the XXI century which are based on quantum foams, loops and holographic features at the Planck scale. In particular, in our approach of 3D DQV, the granular features of the geometry of the fundamental background can be formulated in terms of the following relation

$$g_{\mu\nu}(\vec{x}) = \frac{\hbar^2 c^2}{\beta\Delta\rho_{qvE}^2 V^2}\eta_{\mu\nu} \tag{12}$$

where $\eta_{\mu\nu}$ is the metric of the 3D flat space. On the basis of equation (12), it follows that the features regarding the density of loops of loop quantum gravity (ROVELLI, 2004) can be associated with the quantity $\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}$ depending on the variable energy density as well as the parameter β, and therefore can be seen as the consequence of fundamental processes concerning the virtual particles of the 3D DQV.

The generalized uncertainty relations (11) provide a unifying treatment of microphysics of elementary particles and macrophysics of black holes in terms of a generalized Compton wavelength of the form

$$R_C' = R_S' = \sqrt{\left(\frac{\beta\hbar c}{\Delta\rho_{qvE}V}\right)^2 + \left(\beta l_p^2 \frac{\Delta\rho_{qvE}V}{\hbar c}\right)^2} \quad (13)$$

in a picture where three different regimes (super-Planckian, trans-Planckian and sub-Planckian limits) can be derived as upper manifestations of specific behaviours of the quantum vacuum energy density fluctuations. In light of equation (13), it follows that the crucial connecting loop between microphysics and macrophysics is represented by elementary objects of the Planck scale, which are generated by the geometry of the variable quantum vacuum energy density. These elementary objects are sub-Planckian black holes having the size of the order of the Compton wavelength. Moreover, the generalized uncertainty relations lead also to a unifying treatment of the Casimir effect and cosmological wormholes between two distant regions of the universe, in a picture where the curvature and scale factor of the universe appear as emerging manifestations of the elementary fluctuations of the quantum vacuum energy density as well as of the fluctuating parameter appearing in the generalized uncertainty relations (FISCALETTI and SORLI, 2023).

FROM THE GENERALIZED UNCERTAINTY RELATIONS TO THE MODIFIED RELATIONS REGARDING THE TEMPERATURE AND THE ENTROPY OF SCHWARZSCHILD BLACK HOLES

Black holes can be considered the most mysterious and suggestive objects existing in the universe. As the properties of black holes began to emerge in the 1960s, researchers noted that black holes had some similarities to thermodynamic systems, most notably that any mass falling into a black hole would increase the area of the black hole, and its event horizon. In particular, Hawking showed that black holes must have a real temperature proportional to their surface gravity and an entropy proportional to the square of the black hole's mass and the area of the event horizon and that on the basis of the second law of thermodynamics, their surface areas do not decrease. With these revolutionist premises, today the exploration of some suitable features of black holes, such as Hawking temperature and mass functions, leads to the conclusion that these systems do not violate the first law of thermodynamics. The proportionality of the entropy of black holes with the area of the event horizon would then have led in recent times to the holographic principle, the key capable of opening the access door to quantum gravity, according to which all the information contained in a volume of space is encoded on the surface of that volume. By inextricably unifying the speed of light, Newton's gravitational constant, and Planck's constant, Bekenstein and Hawking showed that the entropy of the universe remains constant or increases because entropy lost from regions outside black holes is always compensated by an equal or greater increase in the black hole's entropy. On the basis of the Bekenstein and Hawking studies, black holes are thermodynamical systems that emit radiation, and thus they must evaporate by means of what is now called Hawking radiation (BARDEEN *et al.*, 1973; BEKENSTEIN, 1973; HAWKING, 1975). In fact, an interesting property of Schwarzschild black holes is that, as a consequence of their negative heat capacity, will lose energy giving origin to the so-called black hole evaporation, even if the final configuration is still an open question because the evaporation of a black hole is associated with a corresponding disappearance of all the information inside the black hole (UNRUH and WALD, 2017; MATUBARO DE SANTI and SANTARELLI, 2019). Possible alternatives to this scenario lie in the possibility that the evaporation of the black hole occurs until its length reaches the Planck scale, or the information results in a final burst, or that black holes quantum tunnels into a white hole, or even that no black hole ever forms (for a review of these perspectives, see, for example, the references UNRUH and WALD, 2017; BIANCHI *et al.*, 2018).

Recently, the issue of finding the quantum corrections to the black hole temperature, entropy, and capacity has received much attention (see, for example, GANGOPADHYAY and DUTTA, 2018; VAGENAS *et al.*, 2018; HASSANABADI *et al.*, 2019b). Here we want to open new scenarios of interpretation of the thermodynamics of black holes (and in particular, Schwarzschild black holes), in the picture of the generalized uncertainty relations of the 3D DQV considered in section 2. In this regard, before all, by starting from the generalized uncertainty relations (11), if one makes the substitution

$$\Delta p \rightarrow \Delta p + \frac{\hbar^2 c^2}{l_p^2 \Delta p} \tag{14}$$

one finds

$$\frac{2M_{ADM}}{r} = \frac{2\Delta p}{c\Delta x} \tag{15}$$

where the quantity

$$M_{ADM} = \frac{\Delta\rho_{qvE} V}{c^2}\left(1 + \frac{\hbar^2 c^2}{\beta l_p^2 \Delta\rho_{qvE}^2 V^2}\right) \tag{16}$$

can be defined as the Arnowitt-Deser-Misner mass in the 3D DQV. The Arnowitt-Deser-Misner mass (16) leads to define a quantum-modified Schwarzschild metric of the form

$$ds^2 = F(r)c^2 dt^2 - F(r)^{-1} dr^2 - r^2 d\Omega^2 \tag{17}$$

where

$$F(r) = 1 - \frac{2G\Delta\rho_{qvE}{}^3 V^3}{M_{Pl}{}^2 c^8 r} - \frac{2\hbar^2 G}{\beta l_p^2 c^2 r \Delta\rho_{qvE} V} \tag{18}$$

The metric (17) – equipped with equation (18) – has important consequences in the sense that it leads to the following expression for the horizon size

$$r_H = R_S' = \frac{2G\Delta\rho_{qvE}{}^3 V^3}{M_{Pl}{}^2 c^8} + \frac{2\hbar^2 G \Delta\rho_{qvE} V}{\beta l_p^2 \Delta\rho_{qvE} V c^2} \tag{19}$$

which, depending on the specific values of the quantum vacuum energy density, becomes

$$r_H \approx \begin{cases} \frac{2G\Delta\rho_{qvE}{}^3 V^3}{M_{Pl}{}^2 c^8} & if\ \Delta\rho_{qvE} V \gg M_{Pl} c^2 \\ \frac{2GM_{Pl}}{c^2}\left(1 + \frac{\hbar^2 c^2}{\beta l_p^2 M_{Pl}^2 c^2}\right) & if\ \Delta\rho_{qvE} V \approx M_{Pl} c^2 \\ \frac{2G\hbar^2}{\beta l_p^2 c^6} & if\ \Delta\rho_{qvE} V \ll M_{Pl} c^2 \end{cases} \tag{20}$$

The first expression of (20), namely

$$r_H \approx \frac{2G\Delta\rho_{qvE}{}^3 V^3}{M_{Pl}{}^2 c^8} \tag{21}$$

regards the super-Planckian regime and corresponds to a regime which coincides with the standard Schwarzschild radius. The intermediate expression regards the trans-Planckian limit, which has a minimum of order l_p. The last expression is linked to the sub-Planckian regime and corresponds to the Compton wavelength.

On the basis of the metric (17), equipped with equation (18), and taking into account the treatment in CARR *et al.* (2015) and CARR (2018), one finds the following results as regards the thermodynamics of the black hole solutions in the three limits considered above (namely super-Planckian, trans-Planckian and sub-Planckian limits (FISCALETTI and SORLI, 2023):

$$T=\frac{M_{Pl}{}^{2}}{8\pi M_{ADM}}\approx\begin{cases}\dfrac{M_{Pl}c^{2}}{8\pi\Delta\rho_{qvE}V\left[1-\beta\left(\frac{M_{Pl}c^{2}}{\Delta\rho_{qvE}V}\right)^{2}\right]} & if\ \Delta\rho_{qvE}V\gg M_{Pl}c^{2}\\ \dfrac{M_{Pl}}{8\pi[1+\beta/2]} & if\ \Delta\rho_{qvE}V\approx M_{Pl}c^{2}\\ \dfrac{\Delta\rho_{qvE}V}{4\pi\beta c^{2}\left[1-\left(\Delta\rho_{qvE}V/M_{Pl}c^{2}\right)^{2}/\beta\right]} & if\ \ \Delta\rho_{qvE}V\ll M_{Pl}c^{2}\end{cases} \tag{22}$$

On the basis of equation (22), the usual Hawking temperature derives from the large $M=\frac{\Delta\rho_{qvE}V}{c^{2}}$ limit while, during the phase of evaporation of the black hole, the temperature reaches a maximum at around T_{Pl} and then decreases to zero as $\frac{\Delta\rho_{qvE}V}{c^{2}}\rightarrow 0$.

In the second part of this section, we want to show how the quantity (21) assumes an important role in determining the properties of a Schwarzschild black hole, and particularly the features of its thermodynamics in the picture of the generalized uncertainty relations (11). In the regime where relation (21) holds, the function (18) appearing in the quantum-modified Schwarzschild metric (17), becomes

$$F(r)=1-\frac{2G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}r} \tag{23}$$

and thus, the metric (17) reads

$$ds^{2}=\left(1-\frac{2G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}r}\right)c^{2}dt^{2}-\left(1-\frac{2G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}r}\right)^{-1}dr^{2}-r^{2}d\Omega^{2} \tag{24}$$

By following PERIVOLAROPOULOS (2017), the quantity (21) leads to the following value for the maximum mass of the black hole

$$M_{max}=\frac{\Delta\rho_{qvE\,max}{}^{3}V^{3}}{4\pi^{2}M_{Pl}{}^{2}c^{6}} \tag{25}$$

By invoking near-horizon geometry considerations, in the generalized uncertainty relation (11) one can set $\Delta x=2\pi r_{H}$, namely

$$\Delta x=\frac{4\pi G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}} \tag{26}$$

Moreover, one can utilize the temperature expression of any massless quantum particle near the horizon of a Schwarzschild black hole

$$T=\frac{c}{k_{B}}\Delta p \tag{27}$$

in order to investigate the Hawking temperature of the black hole. By substituting equations (26) and (27) into the generalized uncertainty relations (11), one thus obtains

$$\frac{4\pi G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{9}}k_{B}T=\frac{\hbar}{2}\left(1+\beta l_{p}^{2}\frac{\Delta\rho_{qvE}^{2}V^{2}}{\hbar^{2}c^{2}}\right) \tag{28}$$

namely

$$T=\frac{\hbar M_{Pl}{}^{2}c^{9}}{8\pi k_{B}G\Delta\rho_{qvE}{}^{3}V^{3}}\left(1+\beta l_{p}^{2}\frac{\Delta\rho_{qvE}^{2}V^{2}}{\hbar^{2}c^{2}}\right) \tag{29}$$

namely

$$T=\frac{M_{Pl}{}^{2}c^{4}}{\Delta\rho_{qvE}{}^{2}V^{2}}T_{0}\left(1+\beta l_{p}^{2}\frac{\Delta\rho_{qvE}^{2}V^{2}}{\hbar^{2}c^{2}}\right) \tag{30}$$

where

$$T_{0}=\frac{\hbar c^{3}}{8\pi k_{B}GM_{bh}} \tag{31}$$

is the ordinary Hawking temperature of the black hole having mass M_{bh}. The quantity (30) can be defined as a modified Hawking temperature of Schwarzschild black holes described by the metric (24), in the picture of the 3D DQV model. The modified Hawking temperature (30) can also be expressed as a function of the maximum mass of the black hole (25):

$$T = \frac{\Delta\rho_{qvE}V}{M_{max}c^2}T_0\left(1+\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}\right) \tag{32}$$

Equation (32) implies that the term $\frac{\Delta\rho_{qvE}V}{M_{max}c^2}$ associated with the quantum-modified Schwarzschild metric, as well as the term $\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}$ associated with the deformation of the background at the Planck scale, modify the standard expression of the Hawking temperature. In particular, it must be remarked that if the following constraint holds

$$\beta l_p^2\frac{\Delta\rho_{qvE}^3V^3}{M_{max}\hbar^2c^4} \to 0 \tag{33}$$

then the Hawking temperature reduces to the value:

$$T = \frac{{M_{Pl}}^2c^4}{{\Delta\rho_{qvE}}^2V^2}T_0 \tag{34}$$

which means that its value is affected by a sort of dilatation determined by the fluctuations of the quantum vacuum energy density. Instead, if the quantum vacuum energy density fluctuations reach the regime of the maximum value of the mass of the black hole (27), namely if

$$\Delta\rho_{qvE}V \to M_{max}c^2 \tag{35}$$

then the Hawking temperature may be written in the form

$$T = T_0\left(1+\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}\right) \tag{36}$$

which, in the regime $\beta \to 0$, reduces to the standard value

$$T = T_0 \tag{37}$$

In the light of equation (36), in the limiting case where the quantum vacuum energy density fluctuations are close to the maximum mass value, the modified Hawking temperature reaches very large values compared to the usual one (while the standard Hawking temperature is recovered in the limit $\beta \to 0$). Instead, in the regime where the quantum vacuum energy density fluctuations are smaller than the critical mass value, then the temperature of the black hole has to be computed by using the general equation (32).

Next, we determine the Schwarzschild black hole entropy by considering the first law of thermodynamics which is defined in the form of:

$$S = V\int\frac{d\Delta\rho_{qvE}}{T} \tag{38}$$

namely, by substituting relation (29):

$$S = V\int\frac{8\pi k_BG{\Delta\rho_{qvE}}^3V^3d(\Delta\rho_{qvE})}{\hbar{M_{Pl}}^2c^9\left(1+\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}\right)} \tag{39}$$

namely

$$S = \frac{8\pi k_BGV^4}{\hbar{M_{Pl}}^2c^9}\int\frac{{\Delta\rho_{qvE}}^3d(\Delta\rho_{qvE})}{\left(1+\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}\right)} \tag{40}$$

By performing the integration, one obtains the following mathematical formulation for the generalized entropy of a Schwarzschld black hole in the 3D quantum vacuum:

namely

$$S = \frac{8\pi k_B G V^2 \hbar}{\beta l_p^2 M_{Pl}{}^2 c^7}\left[\Delta\rho_{qvE}^2 - \frac{\hbar^2 c^2}{\beta l_p^2 V^2} ln\left(\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2} + 1\right)\right] \quad (41)$$

Here, taking account the expression of the Bekenstein-Hawking entropy for the Schwarzschild black hole, given by relation

$$S_0 = 4\pi k_B \frac{M_{bh}{}^2}{M_{Pl}{}^2} \quad (42)$$

equation (41) may also be expressed as follows:

$$S = \frac{2k_B G\hbar}{\beta l_p^2 c^3} S_0 - \frac{k_B G \hbar^3}{\pi \beta^2 l_p^4 M_{Pl}{}^2 c^5} ln\left(\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2} + 1\right) \quad (43)$$

In the light of relation (43), one can say that when $\beta \to 0$ the generalized entropy of the black hole in the 3D DQV, at the second order, may be approximated as

$$S = \frac{2k_B G\hbar}{\beta l_p^2 c^3} S_0 - \frac{k_B G\hbar \Delta\rho_{qvE}^2 V^2}{\pi \beta l_p^2 M_{Pl}{}^2 c^7} + \frac{k_B G}{\pi M_{Pl}{}^2 c^9}\frac{\Delta\rho_{qvE}^4 V^4}{\hbar} \quad (44)$$

The physical meaning of equation (44) is that the value of the generalized entropy of the black hole overcomes important modifications with respect to the standard Bekenstein-Hawking entropy, which are caused by the fluctuations of the quantum vacuum energy density as well as the parameter β appearing in the generalized uncertainty relations. And, according to equation (44), the modified entropy of a Schwarzschild black hole turns out to be characterized by important changes with respect to the standard value also when the fluctuations of the quantum vacuum energy density are negligible. In fact, for $\Delta\rho_{qvE} \to 0$, the entropy (43) becomes

$$S = \frac{2k_B G\hbar}{\beta l_p^2 c^3} S_0 \quad (45)$$

which means that also under the constraint of negligible energy density fluctuations, the entropy of the black hole overcomes a sort of dilatation with respect to the standard Bekenstein-Hawking entropy, with a factor of dilatation given by the quantity $\frac{2k_B G\hbar}{\beta l_p^2 c^3}$ dependent on the parameter describing the deformation of the geometry of the background as well as on fundamental constants of physics.

It is appealing also to formulate the entropy of the black hole (43) in terms of the area of the horizon. In fact, since the area of the horizon, being given by $A = 4l_p \frac{S}{k_B}$, is

$$A = \frac{8G\hbar}{\beta l_p c^3} S_0 - \frac{4G\hbar^3}{\pi \beta^2 l_p^3 M_{Pl}{}^2 c^5} ln\left(\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2} + 1\right) \quad (46)$$

taking account that $A_0 = 16\pi \frac{G^2 M_{bh}{}^2}{c^4}$, by substituting (46) into (43), one obtains

$$S = \frac{k_B \hbar}{2\pi^2 G \beta M_{Pl}^2 l_p^2 c^3} A_0 - \frac{k_B G\hbar^3}{\pi \beta^2 l_p^4 M_{Pl}{}^2 c^5} ln\left(\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2} + 1\right) \quad (47)$$

Here, one can observe that, in the limit $\beta \to 0$, at the first order in a Taylor expansion, relation (47) implies the following formulation of the area theorem in the 3D DQV:

$$S = \frac{k_B \hbar}{2\pi^2 G\beta M_{Pl}^2 l_p^2 c^3} A_0 - \frac{k_B G\hbar \Delta\rho_{qvE}^2 V^2}{\pi\beta l_p^2 {M_{Pl}}^2 c^7} \quad (48)$$

while, for $\Delta\rho_{qvE} \to 0$, one obtains

$$S = \frac{k_B \hbar}{2\pi^2 G\beta M_{Pl}^2 l_p^2 c^3} A_0 \quad (49)$$

Finally, let us see how the heat capacity of the black hole is modified by the generalized uncertainty relations (11). In this regard, by utilizing the following relation

$$C = \left(\frac{1}{V}\frac{dT}{d(\Delta\rho_{qvE})}\right)^{-1} \quad (50)$$

one obtains

$$C = \left(\frac{1}{V}\frac{d\left[\frac{\hbar\pi {M_{Pl}}^2 c^9}{2k_B G {\Delta\rho_{qvE}}^3 V^3}\left(1+\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}\right)\right]}{d[\Delta\rho_{qvE}]}\right)^{-1} \quad (51)$$

namely

$$C = \left(\frac{1}{V}\left[-\frac{3\hbar\pi {M_{Pl}}^2 c^9}{2k_B G {\Delta\rho_{qvE}}^4 V^3}\left(1+\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}\right) + \frac{\pi\beta l_p^2 {M_{Pl}}^2 c^7}{k_B G\hbar {\Delta\rho_{qvE}}^2 V}\right]\right)^{-1} \quad (52)$$

namely

$$C = \left(-\frac{3\hbar\pi {M_{Pl}}^2 c^9}{2k_B G {\Delta\rho_{qvE}}^4 V^4} - \frac{\pi\beta l_p^2 {M_{Pl}}^2 c^7}{2k_B G\hbar {\Delta\rho_{qvE}}^2 V^2}\right)^{-1} \quad (53)$$

namely

$$C = -\frac{2k_B G\hbar {\Delta\rho_{qvE}}^4 V^4}{3\hbar^2\pi {M_{Pl}}^2 c^9 + \pi\beta l_p^2 {M_{Pl}}^2 c^7 {\Delta\rho_{qvE}}^2 V^2} \quad (54)$$

Taking account of the standard expression of the heat capacity in terms of the variable quantum vacuum energy density $C_0 = -8\pi k_B \frac{{\Delta\rho_{qvE}}^2 V^2}{c^4 {M_{Pl}}^2}$, one has

$$C = \frac{G\hbar {\Delta\rho_{qvE}}^2 V^2}{12\hbar^2\pi^2 c^5 + 4\pi^2\beta l_p^2 c^3 {\Delta\rho_{qvE}}^2 V^2} C_0 \quad (55)$$

Equation (55) turns out to be compatible with the fact that the collapse of a black hole ends as soon as the heat capacity function tends to zero. In the light of relation (55), this physical situation occurs when $\Delta\rho_{qvE} \to 0$, which corresponds to the situation of constant mass, called also remnant mass, in the standard interpretation.

EFFECTS OF THE GENERALIZED UNCERTAINTY RELATIONS IN THE THERMODYNAMICS OF SCHWARZSCHILD BLACK HOLES WITH QUINTESSENCE

On the basis of the observational data, we know that the universe is characterized by a phase of accelerated expansion which is associated whit a gravitationally self-repulsive dark energy. In this regard, a possible candidate for dark energy is the so-called quintessence matter, which provides a spatially inhomogeneous component of negative pressure to the cosmological evolution of the universe and is defined by the equation of state

$$P = \omega\rho_{DE} \tag{56}$$

where P is the pressure, ρ_{DE} is the energy density and $-1 < \omega < -1/3$. Since dark energy should compose 70% of the energetic content of the universe, the perspective is opened that quintessence matter is present all over the universe, and thus also around black holes.

The idea that the quantum fluctuations of the background metric of spacetime determine the deformation of a Schwarzschild black hole was originally proposed in a germinal work by KAZAKOV and SOLODUKHIN (1994). Then, in 2003 KISELEV explored the Schwarzschild metric of black holes by considering the action of the quintessence field onto the background. More recently, several authors have analysed the thermodynamics of Schwarzschild black holes and the action of the quintessence associated with their background (in this regard, the literature is very broad, we can cite for example the references KISELEV, 2003; MA *et al.,* 2007; CHEN *et al.,* 2008; VARGHESE and KURIAKOSE, 2009; ZHANG *et al.,* 2009; SALEH *et al.,* 2011; YI-HUAN and ZHONG-HUI, 2011; FERNANDO, 2012, 2013a, 2013b; THOMAS *et al.,* 2012; AZREG-AINOU and RODRIGUES, 2013; THARANATH and KURIAKOSE, 2013; FENG *et al.,* 2014; THARANATH *et al.,* 2014; AZREG-AINOU, 2015; HUSSAIN and ALI, 2015; MALAKOLKALAMI and GHADERI, 2015; GHADERI and MALAKOLKALAMI, 2016; GHOSH, 2016; TOSHMATOV *et al.,* 2017).

By seeking inspiration from KAZAKOV–SOLODUKHIN's work (1994) and KISELEV's approach (2003), NOZARI *et al.* (2020) have recently studied the behaviour of Schwarzschild black holes surrounded by quintessence finding how the quantum fluctuations of the background metric and the quintessence can influence the accretion parameters of the black hole, in the picture of a central 2-dimensional sphere with a radius of the order of the Planck length where the radial component of the 4-velocity and the proper energy density of the accreting fluid turn out to have a finite value. In this section, we want to explore the effects of the generalized uncertainty relations (11) of the 3D DQV model in the thermodynamics of Schwarzschild black holes surrounded by quintessence, in a picture where the quintessence is ultimately associated with more elementary fluctuations of the 3D DQV.

By following the treatment of Luftuoglu, Hamill and Dahbi in LUTFUOGLU *et al.* (2021), a deformed Heisenberg algebra expressed by relation

$$[x, p] = i\hbar[1 + \beta p^2] \tag{57}$$

where β is a small non-negative deformation parameter that is proportional to the Planck length, leads to the following relation between the temperature and the horizon radius of a Schwarzschild black hole surrounded by quintessence:

$$T = \frac{2r_H}{\pi\beta}\left(1 + \frac{3\alpha\omega_q}{r_H^{3\omega_q+1}}\right)\left(1 + \sqrt{1 - \frac{\beta}{4r_H^2}}\right) \tag{58}$$

where ω_q is the quintessential state parameter, and α is the positive normalization factor that depends on the density of quintessence matter. Relation (58) implies that for $\beta = 0$ the temperature of the black hole reduces to the Hawking temperature of Schwarzschild black hole surrounded by the quintessence in the Heisenberg uncertainty principle limit

$$T = \frac{1}{4r_H\pi}\left(1 + \frac{3\alpha\omega_q}{r_H^{3\omega_q+1}}\right) \tag{59}$$

Moreover, one must remark here that, in the regime of $-1 < \omega_q < -1/3$, the quintessence successfully explains the accelerated expansion of the universe.

Now, in our approach of 3D DQV based on the generalized uncertainty relations (11), the perspective is opened to provide a new suggestive re-reading of the thermodynamics of Schwarzschild black holes in the presence of quintessence. In fact, here we take account of the expression (24) regarding the horizon size, and we consider that the parameter β of the

Luftuoglu, Hamill, and Dahbi model is replaced with the quantity $\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}$. Furthermore, above all, on the basis of equation (7), we assume that dark energy and thus quintessence matter do not represent primary physical realities but are the manifestation of more elementary quantum vacuum energy density fluctuations and this means that equation (56) reads

$$P = \frac{35\omega_{qvE} G c^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6 \tag{60}$$

where here ω_{qvE} is interpreted as the state parameter of the 3D DQV describing the fluctuations of the quantum vacuum energy density which determine the negative pressure responsible for the accelerated expansion of space. In other words, in our approach, a black hole surrounded by quintessence means to deal with a black hole where the surrounding region is characterized by quantum vacuum energy density fluctuations mimicking the action of quintessence, thus allowing an explanation of the acceleration of space. As a consequence, our model equation (58) regarding the relation between the temperature and horizon radius of a Schwarzschild black hole is replaced by the following equation expressing the relation between the temperature of a Schwarzschild black hole and the variable energy density

$$T = \frac{4G{\Delta\rho_{qvE}}^3 V^3}{\pi l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2} {M_{Pl}}^2 c^8}\left(1 + \frac{3\alpha\omega_{qvE}}{\left(\frac{2G{\Delta\rho_{qvE}}^3 V^3}{{M_{Pl}}^2 c^8}\right)^{3\omega_{qvE}+1}}\right)\left(1 - \sqrt{1 - \frac{l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}{16\frac{G^2 {\Delta\rho_{qvE}}^6 V^6}{{M_{Pl}}^4 c^{16}}}}\right) \tag{61}$$

namely

$$T = \frac{4G\Delta\rho_{qvE}{}^{\square} V^{\square}\hbar^2 c^2}{\pi l_p^2 {M_{Pl}}^2 c^8}\left(1 + \frac{3\alpha\omega_{qvE}}{\left(\frac{2G{\Delta\rho_{qvE}}^3 V^3}{{M_{Pl}}^2 c^8}\right)^{3\omega_{qvE}+1}}\right)\left(1 - \sqrt{1 - \frac{l_p^2 {M_{Pl}}^4 c^{14}}{16\hbar^2 G^2 {\Delta\rho_{qvE}}^4 V^4}}\right) \tag{62}$$

By following UNRUH and WALD (2017), the generalized uncertainty relations (11) lead to the following constraint regarding the maximal temperature sustained by the black hole:

$$T \le \frac{1}{\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}\left[1 + 3\alpha\omega_{qvE}\left(\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{4\hbar^2 c^2}\right)^{-\frac{3\omega_{qvE}+1}{2}}\right] \tag{63}$$

In the absence of the fluctuations of the quantum vacuum energy density which mimic the action of quintessence matter, one obtains that the temperature of black hole takes values in the range

$$0 \le T \le \frac{1}{2\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}} \tag{64}$$

It must be remarked here that the state parameter ω_{qvE} of the DQV appearing in equations (61)-(63) gives rise to different possible behaviours of the event horizon radius in dependence of their specific value. The parameter ω_{qvE} has the values $-1 < \omega_{qvE} < -1/3$ for a de Sitter horizon which causes the acceleration, and $-1/3 < \omega_{qvE} < 0$ for the asymptotically flat solution. In particular, for $\omega_{qvE} = -2/3$ two event horizon radii emerge given by

$$r_{in} = \frac{1-\sqrt{8M\alpha}}{2\alpha} \tag{65}$$

$$r_{out} = \frac{1+\sqrt{8M\alpha}}{2\alpha} \tag{66}$$

for $\omega_{qvE} = -1/3$ only one horizon radius appears

$$r_H = \frac{2M}{1-\alpha} \tag{67}$$

which, taking account of the general relation (21) regarding the horizon size of a Schwarzschild black hole, implies that this case occurs when the following constraint is satisfied:

$$\frac{2G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}} = \frac{2M}{1-\alpha} \tag{68}$$

namely

$$\Delta\rho_{qvE}{}^{3}V^{3} = \frac{2MM_{Pl}{}^{2}c^{8}}{2G(1-\alpha)} \tag{69}$$

Finally, when $\omega_{qvE} = -1$ one obtains the de Sitter-Schwarzschild solution if the following condition is satisfied

$$\alpha = \frac{\Lambda}{3} \tag{70}$$

where the cosmological constant may be here assimilated to opportune fluctuations of the quantum vacuum energy density on the basis of equation (7).

As a consequence of the peculiar features of the event horizon radius in correspondence with the specific values of the quintessential state parameter, one derives opportune behaviours of the temperature of the Schwarzschild black hole surrounded by vacuum energy density fluctuations mimicking the action of quintessence matter in the 3D DQV model based on the generalized uncertainty relations (11). In particular, for $\omega_{qvE} = -1$, the generalized uncertainty relations (11) determine a modification of the temperature of the Schwarzschild black hole surrounded by these peculiar quantum vacuum energy density fluctuations, given by relation

$$T = \frac{4G\Delta\rho_{qvE}V\hbar^{2}c^{2}}{\pi l_p^2 M_{Pl}{}^{2}c^{8}}\left(1 - 12\alpha\frac{G^{2}\Delta\rho_{qvE}{}^{6}V^{6}}{M_{Pl}{}^{4}c^{16}}\right)\left(1 - \sqrt{1 - \frac{l_p^2 M_{Pl}{}^{4}c^{14}}{16\hbar^{2}G^{2}\Delta\rho_{qvE}{}^{4}V^{4}}}\right) \tag{71}$$

In this situation, the quantum vacuum energy density fluctuations satisfy relation

$$\frac{1}{\sqrt{3\alpha}} \geq \frac{2G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}} \geq \frac{\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}}{2} \tag{72}$$

namely

$$\frac{M_{Pl}{}^{2}c^{7}l_p}{4G\hbar\sqrt{3\alpha}} \geq \Delta\rho_{qvE}{}^{2}V^{2} \geq \sqrt{\beta} \tag{73}$$

and thus, the temperature has values in the range

$$0 \leq T \leq \frac{4-3\alpha\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}{4\pi\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}} \tag{74}$$

Here one can note that, in the regime of negligible fluctuations of the quantum vacuum energy density (and thus in a regime where the generalized uncertainty relations do not assume an important role), the temperature has no superior limit value. However, in general, in the presence of significant fluctuations of the quantum vacuum energy density, the deformation of the geometry of the background leads to the existence of an upper limit value for the temperature, which is just fixed by these fluctuations and by the deformation parameter. These concepts provide a new key to reading the results obtained in (LUTFUOGLU *et al.*, 2021).

Instead, for $\omega_{qvE} = -2/3$, the black hole temperature has the following form

$$T = \frac{4G\Delta\rho_{qvE} V \hbar^2 c^2}{\pi l_p^2 {M_{Pl}}^2 c^8}\left(1 - 4\alpha\frac{G{\Delta\rho_{qvE}}^3 V^3}{{M_{Pl}}^2 c^8}\right)\left(1 - \sqrt{1 - \frac{l_p^2 {M_{Pl}}^4 c^{14}}{16\hbar^2 G^2 {\Delta\rho_{qvE}}^4 V^4}}\right) \tag{75}$$

and therefore, the quantum vacuum energy density fluctuations satisfy relation

$$\frac{{M_{Pl}}^2 c^7 l_p}{8G\hbar\alpha} \geq {\Delta\rho_{qvE}}^2 V^2 \geq \sqrt{\beta} \tag{76}$$

As a consequence, in this situation the temperature has values in the range

$$0 \leq T \leq \frac{1-\alpha\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}{\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}} \tag{77}$$

Coherently with the treatment made in LUTFUOGLU *et al.* (2021), here we find that the 3D DQV based on the generalized uncertainty relations (11) predicts the same upper limit, 1, for the horizon size, of the Heisenberg uncertainty principle, but now here we can provide an explanation of this result in terms of more fundamental properties of the 3D DQV. Moreover, according to equation (77), the temperature has a maximum value determined by the behaviour of the variable quantum vacuum energy density.

On the other hand, for $\omega_{qvE} = -1/3$, the black hole temperature has the following form

$$T = \frac{4G\Delta\rho_{qvE} V \hbar^2 c^2}{\pi l_p^2 {M_{Pl}}^2 c^8}(1 - \alpha)\left(1 - \sqrt{1 - \frac{l_p^2 {M_{Pl}}^4 c^{14}}{16\hbar^2 G^2 {\Delta\rho_{qvE}}^4 V^4}}\right) \tag{78}$$

In this case, the horizon size is not characterized by an upper bound, while the temperature has values in the range

$$0 \leq T \leq \frac{1-\alpha}{\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}} \tag{79}$$

On the basis of the formalism (79), a maximum value of the temperature is determined by behaviour of the variable energy density of the 3D DQV, which thus provides a new suggestive key of reading of the results obtained in LUTFUOGLU *et al.* (2021).

Let us analyse now the behaviour of the heat capacity. In this regard, one obtains that a Schwarzschild black hole surrounded by the specific quantum vacuum energy density fluctuations mimicking the action of quintessence is characterized by a heat capacity given by relation

$$C = -\frac{\pi\beta l_p^2 \Delta\rho_{qvE}^2 V^2}{4\hbar^2 c^2}\frac{\left(1+\frac{3\alpha\omega_{qvE}}{\left(\frac{2G{\Delta\rho_{qvE}}^3 V^3}{{M_{Pl}}^2 c^8}\right)^{3\omega_{qvE}+1}}\right)\sqrt{1-\frac{l_p^2 {M_{Pl}}^4 c^{14}}{16\hbar^2 G^2 {\Delta\rho_{qvE}}^4 V^4}}}{\left(1-\frac{9\alpha{\omega_{qvE}}^2}{\left(\frac{2G{\Delta\rho_{qvE}}^3 V^3}{{M_{Pl}}^2 c^8}\right)^{3\omega_{qvE}+1}}\right)\left(1-\sqrt{1-\frac{l_p^2 {M_{Pl}}^4 c^{14}}{16\hbar^2 G^2 {\Delta\rho_{qvE}}^4 V^4}}\right)+\frac{l_p^2 {M_{Pl}}^4 c^{14}}{16\hbar^2 G^2 {\Delta\rho_{qvE}}^4 V^4}\frac{3\alpha\omega_q\left(1+3\alpha\omega_{qvE}\right)}{\left(\frac{2G{\Delta\rho_{qvE}}^3 V^3}{{M_{Pl}}^2 c^8}\right)^{3\omega_{qvE}+1}}} \tag{80}$$

which tends to zero thus yielding the black hole remnant under the constraint

$$\frac{2G\Delta\rho_{qvE}{}^{3}V^{3}}{M_{Pl}{}^{2}c^{8}} = \frac{1}{2}\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}} \tag{81}$$

namely

$$\Delta\rho_{qvE}{}^{4}V^{4} = \beta l_p^2 \frac{M_{Pl}{}^{4}c^{16}}{16G^2\hbar^2 c^2} \tag{82}$$

Now, by inserting equation (82) into equation (61), we get the non-zero black hole remnant temperature

$$T_{rem} = \frac{1}{\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}\left(1 + 3\alpha\omega_{qvE}\left(\frac{2}{\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}\right)^{3\omega_{qvE}+1}\right) \tag{83}$$

with a corresponding mass

$$M_{rem} = \frac{\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}{4}\left(1 - \alpha\left(\frac{2}{\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}\right)^{3\omega_{qvE}+1}\right) \tag{84}$$

In the light of equations (80) and (83), in the absence of the fluctuations of the quantum vacuum energy density which mimic the action of the quintessence matter, the heat capacity and the remnant temperature respectively become

$$C = -\frac{\pi\beta l_p^2 \Delta\rho_{qvE}^2 V^2}{4\hbar^2 c^2}\frac{\sqrt{1-\frac{l_p^2 M_{Pl}{}^{4}c^{14}}{16\hbar^2 G^2 \Delta\rho_{qvE}{}^{4}V^4}}}{1-\sqrt{1-\frac{l_p^2 M_{Pl}{}^{4}c^{14}}{16\hbar^2 G^2 \Delta\rho_{qvE}{}^{4}V^4}}} \tag{85}$$

$$T_{rem} = \frac{1}{\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}} \tag{86}$$

Now, with these results at hand, we want to explore their consequences as regards the behaviour of the important thermodynamic properties of a Schwarzschild black hole, for the opportune values of the state parameter of the 3D DQV in the range $-1 < \omega_{qvE} < -1/3$, which correspond to the opportune cosmological situations of the event horizon radius, above mentioned. In this regard, for $\omega_{qvE} = -1$, the heat capacity, remnant temperature, and remnant mass of the Schwarzschild black hole whose surrounding region is characterized by quantum vacuum energy density fluctuations mimicking the action of quintessence matter, become

$$C = -\frac{\pi\beta l_p^2 \Delta\rho_{qvE}^2 V^2}{4\hbar^2 c^2}\frac{\left(1-12\alpha\frac{G^2\Delta\rho_{qvE}{}^{6}V^6}{M_{Pl}{}^{4}c^{16}}\right)\sqrt{1-\frac{l_p^2 M_{Pl}{}^{4}c^{14}}{16\hbar^2 G^2 \Delta\rho_{qvE}{}^{4}V^4}}}{\left(1-36\alpha\frac{G^2\Delta\rho_{qvE}{}^{6}V^6}{M_{Pl}{}^{4}c^{16}}\right)\left(1-\sqrt{1-\frac{l_p^2 M_{Pl}{}^{4}c^{14}}{16\hbar^2 G^2 \Delta\rho_{qvE}{}^{4}V^4}}\right)+\frac{3\alpha\beta l_p^2\frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}{2}} \tag{87}$$

$$T_{rem} = \frac{1}{\pi\sqrt{\beta l_p^2 \frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}}\left(1 - \frac{3\alpha\beta l_p^2\frac{\Delta\rho_{qvE}^2 V^2}{\hbar^2 c^2}}{4}\right) \tag{88}$$

$$M_{rem}=\frac{\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}}{4}\left(1-\frac{\alpha\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}{4}\right) \tag{89}$$

Instead, for $\omega_{qvE}=-2/3$, the heat capacity, remnant temperature and remnant mass of the Schwarzschild black hole surrounded by these peculiar quantum vacuum energy density fluctuations mimicking the action of quintessence matter, become

$$C=-\frac{\pi\beta l_p^2\Delta\rho_{qvE}^2V^2}{4\hbar^2c^2}\frac{\left(1-4\alpha\frac{G\Delta\rho_{qvE}{}^3V^3}{M_{Pl}{}^2c^8}\right)\sqrt{1-\frac{l_p^2M_{Pl}{}^4c^{14}}{16\hbar^2G^2\Delta\rho_{qvE}{}^4V^4}}}{\left(1-8\alpha\frac{G\Delta\rho_{qvE}{}^3V^3}{M_{Pl}{}^2c^8}\right)\left(1-\sqrt{1-\frac{l_p^2M_{Pl}{}^4c^{14}}{16\hbar^2G^2\Delta\rho_{qvE}{}^4V^4}}\right)+\frac{\alpha\beta l_p^2M_{Pl}{}^2c^8}{4G\hbar^2c^2\Delta\rho_{qvE}{}^{\square}V^{\square}}} \tag{90}$$

$$T_{rem}=\frac{1-\alpha\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}}{\pi\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}} \tag{91}$$

$$M_{rem}=\frac{\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}}{4}\left(1-\frac{\alpha\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}}{2}\right) \tag{92}$$

Finally, for $\omega_{qvE}=-1/3$, one obtains the following results regarding the heat capacity, the remnant temperature and the remnant mass of the black hole:

$$C=-\frac{\pi\beta l_p^2\Delta\rho_{qvE}^2V^2}{4\hbar^2c^2}\frac{\sqrt{1-\frac{l_p^2M_{Pl}{}^4c^{14}}{16\hbar^2G^2\Delta\rho_{qvE}{}^4V^4}}}{1-\sqrt{1-\frac{l_p^2M_{Pl}{}^4c^{14}}{16\hbar^2G^2\Delta\rho_{qvE}{}^4V^4}}} \tag{93}$$

$$T_{rem}=\frac{1-\alpha}{\pi\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}} \tag{94}$$

$$M_{rem}=\frac{(1-\alpha)\sqrt{\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{\hbar^2c^2}}}{4} \tag{95}$$

All equations (87)-(95) show that the peculiar quantum vacuum energy density fluctuations are the fundamental entities that generate the different behaviours of heat capacity, remnant temperature, and remnant mass of a Schwarzschild black hole, for the opportune values of the state parameter of the 3D DQV in the range $-1<\omega_{qvE}<-1/3$, corresponding to opportune cosmological situations of the event horizon radius.

Moreover, by introducing the relation between the temperature of a Schwarzschild black hole and the variable energy density (namely equation (62)) inside the customary definition of the entropy $S=\int\frac{dM}{T}$, we can derive the expression of the entropy of the black hole in the presence of the peculiar fluctuations of the quantum vacuum energy density mimicking the action of the quintessence matter:

$$S=\frac{2\pi G^2\Delta\rho_{qvE}{}^6V^6}{M_{Pl}{}^4c^{16}}\left(1+\sqrt{1-\frac{l_p^2M_{Pl}{}^4c^{14}}{16\hbar^2G^2\Delta\rho_{qvE}{}^4V^4}}\right)-\pi\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{8\hbar^2c^2}\ln\frac{2G\Delta\rho_{qvE}{}^3V^3}{M_{Pl}{}^2c^8}-$$
$$\pi\beta l_p^2\frac{\Delta\rho_{qvE}^2V^2}{8\hbar^2c^2}\ln\left(1+\sqrt{1-\frac{l_p^2M_{Pl}{}^4c^{14}}{16\hbar^2G^2\Delta\rho_{qvE}{}^4V^4}}\right) \tag{96}$$

We can observe here that the entropy of the black hole is not influenced by the surrounding quantum vacuum energy density fluctuations mimicking the action of quintessence, while it is modified by the generalized uncertainty relations describing the deformation of the geometry of the background, in an analogous way to what happens in absence of these peculiar vacuum fluctuations, as we have seen in the previous section. Moreover, in the Heisenberg uncertainty principle limit, the entropy assumes the familiar expression, $S = \pi r_H^2 \; = \; A/4$. Finally, in our approach, we can also express the mass as a function of the quantum vacuum energy density fluctuations mimicking the action of quintessence and the horizon radius:

$$M = \frac{r_H}{2} + \frac{35Gc^2 r_H^3}{6\pi\hbar^4 V \omega_{qvE}} \left(\frac{V}{c^2} \Delta\rho_{qvE}^{DE}\right)^6 \tag{97}$$

and, by inserting (97) and the following expression of the volume in terms of the horizon radius and state parameter of the vacuum

$$V = \frac{r_H^3}{3\omega_{qvE}^2} \tag{98}$$

inside equation (62), one finds

$$T = \frac{2\left(3\omega_{qvE}^2 V\right)^{1/3}}{\pi\beta} \left(1 + 2\frac{35Gc^2}{6\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6 \left(3\omega_{qvE}^2 V\right)^{2/3}\right)\left(1 - \sqrt{1 - \frac{\beta}{4\left(3\omega_{qvE}^2 V\right)^{2/3}}}\right) \tag{99}$$

On the basis of equation (99), for $T = 1$ isotherm, one can obtain the modified equation of state of the Schwarzschild black hole in the 3D DQV in the form

$$\frac{35\omega_{qvE} G c^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6 = \frac{\omega_{qvE}}{2\left(3\omega_{qvE}^2 V\right)^{2/3}}\left[1 - \frac{\pi\beta}{2\left(3\omega_{qvE}^2 V\right)^{1/3}} \frac{1}{1 - \sqrt{1 - \frac{\beta}{4\left(3\omega_{qvE}^2 V\right)^{2/3}}}}\right] \tag{100}$$

namely

$$\left(\Delta\rho_{qvE}^{DE}\right)^6 = \frac{\pi\hbar^4 c^{10}}{35GV^5\left(3\omega_{qvE}^2 V\right)^{2/3}}\left[1 - \frac{\pi\beta}{2\left(3\omega_{qvE}^2 V\right)^{1/3}} \frac{1}{1 - \sqrt{1 - \frac{\beta}{4\left(3\omega_{qvE}^2 V\right)^{2/3}}}}\right] \tag{101}$$

Equation (101) expresses the link of the peculiar fluctuations of the variable energy density of the 3D DQV which are responsible for the acceleration of space, with the fundamental parameters describing the geometry of the 3D DQV as well as with fundamental constants.

CONCLUSIONS AND OPEN PERSPECTIVES

In the model of the three-dimensional dynamic quantum vacuum characterized by a variable energy density, one deals with a deformation of the geometry at the Planck scale expressed by opportune generalized uncertainty relations (11) which introduces new scenarios of re-reading of the thermodynamics of Schwarzschild black holes. In particular, they lead to a modification of the standard expression of the Hawking temperature as well as to a generalized entropy of the black hole which overcomes a sort of dilatation with respect to the standard Bekenstein-Hawking entropy and a modified heat capacity which implies that the physical situation represented by the remnant mass in the standard interpretation (which correspond to the ending of the collapse of the black hole) is obtained under the constraint of absence of quantum vacuum energy density fluctuations.

Moreover, the behaviour of Schwarzschild black holes surrounded by quintessence is explained in terms of negative pressure (60) ultimately associated with more fundamental specific quantum vacuum energy density fluctuations which are responsible for the accelerated expansion of the universe. In other words, in the three-dimensional dynamic quantum vacuum, whose geometry is ruled by the generalized uncertainty relations (11), dark energy does not exist as a primary physical reality but is the manifestation of more elementary quantum vacuum energy density fluctuations as well as of a peculiar state parameter characterizing them. We have thus explored that heat capacity, remnant temperature, and remnant mass of the Schwarzschild black hole whose surrounding region is characterized by quantum vacuum energy density fluctuations mimicking the action of quintessence matter, turn out to have values that are determined by the fluctuations of the quantum vacuum energy density as well as the deformation parameter, thus providing an explanation of the results of other current research in terms of a more fundamental background. In particular, the temperature of black holes under opportune constraints regarding the state parameter of the dynamic quantum vacuum which corresponds to specific features of the horizon is explored and we have found that, in each peculiar range of the state parameter, it has a maximum value which is directly associated with a specific behaviour of the quantum vacuum energy density fluctuations. Finally, an equation of state expressing the link between the peculiar fluctuations of the variable energy density of the 3D DQV which is responsible for the acceleration of space, and the fundamental parameters describing the geometry of the 3D DQV, is obtained. In this regard, the perspective is opened that this equation of state regarding the behaviour of the dark energy density fluctuations introduces new keys for re-reading of crucial cosmological issues and this could be a relevant issue to be explored in future works.

On the other hand, another important step regards the possibility of finding a relation of the model here proposed with other relevant alternative approaches which invoke the existence of minimal lengths and with current models of thermodynamics of black holes surrounded by quintessence matter. In this regard, in particular, GHOSH *et al.*, (2018) have recently explored the thermodynamics of black holes in D-dimensional Lovelock gravity, where the action contains higher order curvature terms and reduces to the Einstein-Hilbert action in four dimensions, finding that mass, entropy and temperature are modified to the surrounding quintessence background owing to the quintessence energy density term

$$\rho_q = -\frac{\omega_q q(D-1)(D-2)}{2r^{(D-1)(\omega_q+1)}} \tag{102}$$

where q is appropriately chosen such that $\omega_q \leq 0$. In our approach, on the basis of equation (60), the quintessence matter is replaced by the opportune specific fluctuations of the energy density of the three-dimensional quantum vacuum which appears in equation (7). Moreover, the quintessential state parameter ω_q is replaced by the state parameter of the three-dimensional quantum vacuum ω_{qvE} which describes the specific fluctuations responsible for the accelerated expansion of the universe. Therefore, taking account of equations (7) and (60), we obtain

$$\frac{35Gc^2}{2\pi\hbar^4 V}\left(\frac{V}{c^2}\Delta\rho_{qvE}^{DE}\right)^6 = -\frac{\omega_q q(D-1)(D-2)}{2r^{(D-1)(\omega_q+1)}} \tag{103}$$

namely

$$\left(\Delta\rho_{qvE}^{DE}\right)^6 = -\frac{\pi\hbar^4 c^{10} V^5 \omega_{qvE} q(D-1)(D-2)}{35Gr^{(D-1)(\omega_q+1)}} \tag{104}$$

Equation (104) implies therefore

$$\frac{q(D-1)(D-2)}{r^{(D-1)(\omega_q+1)}} = -\frac{35G}{\pi\hbar^4 c^{10} V^5 \omega_{qvE}}\left(\Delta\rho_{qvE}^{DE}\right)^6 \tag{105}$$

which means that the action of the higher D-dimensional background in the thermodynamics of the black hole in Lovelock gravity can be seen as a secondary effect of a more fundamental primordial variable quantum vacuum energy density.

Finally, a suggestive problem would be to find an explanation of black hole mass growth due to cosmological coupling, which has been recently tested in elliptical galaxies over redshift satisfying relation $0 < z \leq 2{,}5$ (FARRAH *et al.*, 2023). In this regard, a parametrization of the variation of the black hole mass in time has been proposed by CROKER *et al.*, (2019) as

$$M(a) = M(a_i)\left(\frac{a}{a_i}\right)^k \tag{106}$$

where a_i is the scale factor at which the black hole becomes cosmologically coupled and $k \approx 3$ is the maximum value of the cosmological coupling strength which is associated with positive energy density (CROKER *et al.*, 2021) and it has been proposed that stellar remnant $k = 3$ black holes are the astrophysical origin of late-time accelerating expansion of the universe (FARRAH *et al.*, 2023). In the model of 3D DQV based on the generalized uncertainty relations (11), the key point regarding the remnant mass of the black holes is represented by equation (84), and therefore equation (106) here becomes

$$M_{rem}(\Delta\rho_{qvE}) = \frac{\sqrt{\beta l_p^2 \frac{\Delta\rho_{coupled}^2 V^2}{\hbar^2 c^2}}}{4}\left(1 - \alpha\left(\frac{2}{\sqrt{\beta l_p^2 \frac{\Delta\rho_{coupled}^2 V^2}{\hbar^2 c^2}}}\right)^{3\omega_{qvE}+1}\right)\left(\frac{\Delta\rho_{qvE}}{\Delta\rho_{coupled}}\right)^k \tag{107}$$

where $\Delta\rho_{coupled}$ denotes the quantum vacuum energy density fluctuations at which the black hole becomes cosmologically coupled and $k \approx 3$ is the cosmological coupling strength. As regards the consequences of relation (105) as regards the link of the D-dimensional background of Lovelock gravity and the corresponding more fundamental parameters of the three-dimensional dynamic quantum vacuum, as well as the implications of equation (107) in the description of the formation of black holes and explanation of cosmological black hole mass changes, further research will give you more information.

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