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NONCOMMUTATIVE SCALAR FIELDS: QUANTUM SYMMETRIES AND BRAIDED BV QUANTIZATION

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ABSTRACT. It is strongly believed that the fully consistent quantum gravity theory should lead to a quantum spacetime. The continuous description of spacetime in terms of differential manifolds is no longer adequate at the quantum gravity energies. Although the full quantum gravity is still unknown, there are several physical theories/models that incorporate the idea of a quantum spacetime. Some of them are: string theory, loop quantum gravity, noncommutative (NC) geometry. In this short paper we address the properties of scalar fields on noncommutative spacetimes. We will discuss the deformed symmetries and quantization of the ϕ^4-theory in four dimensions for two different NC deformations: Moyal or θ-constant NC spacetime and the λ-Minkowski spacetime. Using the newly developed braided BV quantization, we will show that there are no non-planar diagrams and no UV/IR mixing for both NC deformations. However, a nontrivial deformation of the momentum conservation law appears in the λ-Minkowski spacetime.

Keywords: NC scalar field, Drinfel'd twist, braided L_∞-algebra, BV quantization

INTRODUCTION

Following the successful formulation of Quantum Mechanics (QM) and the growing experimental results in particle physics, in the late 1920s the development of Quantum Field Theory (QFT) began. In the early days of Quantum Electrodynamics, the problem of calculating electron self-energy appeared: the UV limit of the obtained results was divergent. Motivated by the success of his uncertainty relations, Heisenberg (HEISENBERG, 1930) suggested imposing noncommutativity between coordinate operators

$$[x_i, x_j] \sim \theta^{ij} , \tag{1.1}$$

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resulting in the uncertainty relations between coordinates

$$\Delta x_i \, \Delta x_j \; \geq \theta^{ij} \, . \tag{1.2}$$

However, Heisenberg soon gave up this idea, regarding it as too radical. In the attempt to eliminate the ultra-violet (UV) divergences in QFT, in 1947, Snyder proposed a way to obtain a discrete space-time replacing the usual coordinates by the operators satisfying nontrivial commutation relations (SNYDER, 1947). This was the first time that noncommutative (NC) spaces appeared in physics. However, Snyder's idea was not accepted at that time. One reason was that the renormalization theory came out to be very successful in eliminating divergences in QFT. The second reason was the mathematical complexity of NC spaces. It took some time until the mathematical structure was formulated, and the first physical models were derived. The mathematical structure of NC spaces became clearer in the 1980s and the 1990s. One of the main results was the Gelfand-Naimark theorem (GELFAND and NAIMARK, 1947). It states that it is possible to describe a manifold by an (appropriately restricted) algebra of functions on the manifold. The space behind can be ignored completely since all the important information are now contained in the algebra of functions. This theorem can be generalized in different ways. For example, the algebra of functions does not have to be commutative, it can be a deformation of the commutative one. If the deformation is continuous, then there exists a set of continuous parameters that control the noncommutativity. The usual commutative space-time (manifold) is obtained for special values of these parameters. The deformed algebra of functions is not the algebra of functions on a manifold but on a "noncommutative space". The main notion that is lost in this generalization is that of a point: Noncommutative geometry is pointless geometry.

In this short paper we address the properties of scalar fields on noncommutative spacetimes using the formalism of the Drinfel'd twist and the homological perturbation theory. In the next section we review the construction of NC ϕ^4-theory scalar field theory via the Drinfel'd twist and the briaded L_∞-algebra. In Sections 3 and 4 we then discuss the deformed symmetries and quantization of the ϕ^4-theory in four dimensions for two different NC deformations: Moyal or θ-constant NC spacetime and the λ-Minkowski spacetime respectfully.

Noncommutative scalar field theory

Nowadays there are different approaches to noncommutative geometry (CONNES, 1994, LANDI, 1997; MADORE, 1999; ASCHIERI *et al.*, 2008). A well-defined way to introduce noncommutative deformations of spacetime and the corresponding symmetries (Lorentz, Poincaré, conformal...) is via Drinfel'd twist (DRINFEL'D, 1985). The twist deforms the symmetry (Hopf) algebra by changing the comultiplication, that is the Leibniz rule of the symmetry generators when they act on products of representations changes. In this way some of the standard conservation laws become deformed. To have modules algebras (functions, differential forms, tensors...) consistent with the deformed symmetries, the twist operator has to be applied to deform them as well. In particular, the new multiplication in module algebras, ⋆-multiplication, is introduced via the inverse of the given twist operator.

Abelian twist – In this paper we will focus on a particular class of Drinfel'd twist operators, the Abelian Drinfel'd twists. They are defined using the set of commuting vector fields

$$\mathcal{F} = \exp\left(-\frac{\mathrm{i}\theta^{AB}}{2}(X_A \otimes X_B)\right) . \tag{2.1}$$

The set of commuting vector fields X_A, X_B are chosen between the generators of the Poincaré algebra. The twist operator can in generally be constructed from a more general set of vector fields. Our choice ensures that the standard Poincaré symmetry of (quantum field theory on) Minkowski spacetime is replaced by a deformed/twisted Poincaré symmetry. The matrix θ^{AB} is antisymmetric and constant. It is considered as a small deformation parameter[2].

The pointwise product between functions $\mu(f \otimes g) = f \cdot g$ is replaced by the noncomutative $\star$-product, defined via

$$\begin{aligned} f \star g &= \mu \circ \mathcal{F}^{-1}(f \otimes g) \\ &= f \cdot g + \frac{\mathrm{i}\theta^{AB}}{2}(X_A \mathrm{f} \cdot X_B g) + \boldsymbol{O}(\theta^2)\,. \end{aligned} \tag{2.2}$$

The noncommutativity in the space of functions is expressed via triangular $\mathcal{R}$ matrix so that the following relations hold

$$\begin{aligned} f \star g &= \mathcal{R}^{-1}(g \star f) = \mathrm{R}_\alpha g \star \mathrm{R}^\alpha f\,, \\ \mathcal{R}^{-1} &= \mathcal{F}^2 \equiv \mathrm{R}_\alpha \otimes \mathrm{R}^\alpha\,. \end{aligned} \tag{2.3}$$

To define a scalar field theory on the NC spacetime defined by (2.2) and (2.3), we use the formalism of the braided L_∞-algebras. It is well known that the data of any classical field theory are completely encoded in a corresponding L_∞-algebra (HOHM and ZWIEBACH, 2017; JURČO *et al*., 2019). Applying the twist formalism to a classical L_∞-algebra results in the braided L_∞-algebra and the corresponding field theory is referred to as the braided field theory (DIMITRIJEVIĆ ĆIRIĆ *et al*., 2021).

Classical braided scalar field theory – We illustrate this formalism on the simple example of a real massive scalar field ϕ in four dimensions with $\lambda\,\phi^4$-interaction. The underlying braided L_∞-algebra of this theory is graded vector space $V = V_1 \oplus V_2$, consisting of the space of fields V_1 and the space of antifields V_2, equipped with the set of braided brackets

$$\begin{aligned} \ell_1^\star(\phi) &= (-\Box - m^2)\,\phi\,, \\ \ell_3^\star(\phi_1, \phi_2, \phi_3) &= \lambda\,\phi_1 \star \phi_2 \star \phi_3\,, \\ &= \lambda \mathrm{R}_\alpha \phi_2 \star \mathrm{R}^\alpha \phi_1 \star \phi_3 = \phi_1 \star \mathrm{R}_\alpha \phi_3 \star \mathrm{R}^\alpha \phi_2\,, \end{aligned} \tag{2.5}$$

for $\lambda \in \mathbb{R}$ and all $\phi_i \in V_1$. In the last line of (2.5) we explicitly wrote the braided symmetry of the $\ell_3^\star$ bracket.

The cyclic pairing of degree -3 is taken to be

$$\langle \phi, \phi^+ \rangle_\star = \int \mathrm{d}^4\mathrm{x}\;\phi \star \phi^+ \tag{2.6}$$

where $\phi \in V_1$ and $\phi^+ \in V_2$.

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[2]More precisely, the twist (2.1) is defined as

$$\mathcal{F} = \exp\left(-\frac{\mathrm{i}\kappa\theta^{AB}}{2}(X_A \otimes X_B\right)$$

where θ^{AB} is the constant antisymmetric matrix and the small deformation parameter is labeled with κ. To shorten the notation, κ is usually absorbed in θ^{AB} and we say that θ^{AB} is the small deformation parameter.

This construction leads to the action functional

$$\begin{aligned} S^{\star}(\phi) &= \frac{1}{2!}\langle \phi, \ell_1^{\star}(\phi)\rangle_{\star} - \frac{1}{3!}\langle \phi, \ell_2^{\star}(\phi,\phi)\rangle_{\star} - \frac{1}{4!}\langle \phi, \ell_3^{\star}(\phi,\phi,\phi)\rangle_{\star} + \cdots \\ &= \int d^4x \;\; \frac{1}{2}\phi(-\Box - m^2)\phi - \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \;, \end{aligned} \tag{2.7}$$

and the equation of motion

$$\begin{aligned} F_{\phi} &= \ell_1^{\star}(\phi) - \frac{1}{2}\,\ell_2^{\star}(\phi,\phi) - \frac{1}{4!}\,\ell_3^{\star}(\phi,\phi,\phi) + \cdots = 0 \\ 0 &= (-\Box - m^2)\,\phi - \frac{\lambda}{4!}\,\phi \star \phi \star \phi \,. \end{aligned} \tag{2.8}$$

The free theory remains undeformed, while the interaction term has a NC contribution.

To better understand the constructed theory, we will choose two different Abelian twists based on the Poincaré algebra of the four dimensional Minkowski spacetime. In the next section we work with the Moyal (θ-constant deformation), while in Section 4 we discuss the λ-Minkowski spacetime.

Braided BV quantization: Moyal spacetime

The Moyal twist is defined as

$$\mathcal{F} = \exp\big(-\,\mathrm{i}/2\;\theta^{ij}\;\partial_i \otimes \partial_j\big)\;, \tag{3.1}$$

where $\big(\theta^{ij}\big)$ is a $(3)\times(3)$ antisymmetric real-valued matrix, and $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1,2,3$ are vector fields generating spatial translations on $\mathbb{R}^{1,3}$. In order to simplify some of the analysis in the quantum field theory, such as the treatment of time-ordering, as well as avoiding potential issues with unitarity, we restricted the twist (2.1) to spatial translation only. The corresponding $\star$-product of functions and the $\mathcal{R}$-matrix are given by

$$\begin{aligned} f \star g &= \mu \circ \mathcal{F}^{-1}\,(f \otimes g) \\ &= f \cdot g + \frac{\mathrm{i}\theta^{ij}}{2}\big(\partial_i f \cdot \partial_j g\big) + \boldsymbol{O}(\theta^2)\,, \\ \mathcal{R} &= \mathrm{R}^{\alpha} \otimes \mathrm{R}_{\alpha} = \mathcal{F}^{-2} = \exp\big(\mathrm{i}\theta^{ij}\;\partial_i \otimes \partial_j\big)\,. \end{aligned} \tag{3.2}$$

Braided scalar QFT – Let us now explain how to compute correlation functions of the interacting braided scalar field theory using the braided BV formalism developed in (NGUYEN *et al.*, 2021). More details can be found in (BOGDANOVIĆ *et al.*, 2024).

We start from the cohomology $H^{\bullet}(V)$ of the abelian L_{∞}-algebra (V, ℓ_1), which describes the classical vacua of the free (braided) scalar field theory on $\mathbb{R}^{1,3}$. The cohomology $H^{\bullet}(V)$ is also an abelian L_{∞}-algebra concentrated in degrees 1 and 2, given by the solution space $H^1(V) = \ker(\ell_1)$ of the massive Klein–Gordon equation $(\Box + m^2)\,\phi = 0$ and the space $H^2(V) = \operatorname{coker}(\ell_1)$ with the trivial differential 0.

The n-point correlation functions in momentum space are defined as

$$\tilde{G}_n^{\star}(p_1, \dots, p_n)^{\text{int}} = \sum_{m=1}^{\infty} \mathrm{P}((\mathrm{i}\hbar\Delta_{\text{BV}}\mathrm{H} + \{S_{\text{int}}, -\}_{\star}\mathrm{H})^m(\mathrm{e}^{p_1} \odot_{\star} \cdots \odot_{\star} \mathrm{e}^{p_n})) \quad . \tag{3.3}$$

This is a formal power series expansion in $\hbar$ and the coupling constant λ in S_{int} . Let us clarify the operators appearing in this definition.

We start by defining a translation-equivariant projection $\mathrm{p} : \mathrm{V} \to H^{\bullet}(V)$ of degree 0 and a translation-invariant contracting homotopy $\mathrm{h} : V_2 \to V_1$. For this, let $\mathsf{G} : C^{\infty}(\mathbb{R}^{1,3}) \to C^{\infty}(\mathbb{R}^{1,3})$ denote the scalar Feynman propagator

$$\mathsf{G} = -\frac{1}{\Box + m^2} \qquad \text{with} \qquad \widetilde{\mathsf{G}}(k) = \frac{1}{k^2 - m^2} \, , \tag{3.4}$$

where $\widetilde{\mathsf{G}}(k)$ are the eigenvalues of the Green operator G when acting on plane wave eigenfunctions of the form $\mathrm{e}^{\,\mathrm{i}\, k \cdot x}$. It satisfies

$$\ell_1 \circ \mathsf{G} = -(\Box + m^2) \circ \mathsf{G} = \mathrm{id}_{C^{\infty}(\mathbb{R}^{1,3})} \, . \tag{3.5}$$

ince we are interested in calculating (braided) correlation functions, we take the trivial projection map $\mathrm{p} = 0$, or more accurately we restrict the cochain complex of $H^{\bullet}(V)$ to its trivial subspaces. With these choices, the contracting homotopy $\mathrm{h} : V_2 \to V_1$ is given by the propagator $\mathrm{h} = \mathsf{G}$. In momentum space representation the contracting homotopy acts as

$$\mathrm{h}(\phi^{+})(k) = \frac{\phi^{+}(k)}{k^2 - m^2} \, . \tag{3.6}$$

Next, we extend the maps p and h to the braided space of functionals $\mathrm{Sym}_{\mathcal{R}}(V[2])$ on V. This space represents the space of obesrvables in QFT. The data given above induce a trivial projection map $\mathsf{P} : \mathrm{Sym}_{\mathcal{R}}(V[2]) \to \mathrm{Sym}_{\mathcal{R}}\big(H^{\bullet}(V[2])\big)$ given by

$$\mathsf{P}(1) = 1 \qquad \text{and} \qquad \mathsf{P}\Big(\varphi_1 \odot_{\star} \cdots \odot_{\star} \varphi_n\Big) = 0 \, . \tag{3.7}$$

The extended contracting homotopy $\mathsf{H} : \mathrm{Sym}_{\mathcal{R}}(V[2]) \to \mathrm{Sym}_{\mathcal{R}}(V[2])$ is defined as

$$\begin{aligned} \mathrm{H}(1) &= 0 \, , \\ H\Big(\varphi_1 \odot_{\star} \cdots \odot_{\star} \varphi_n\Big) &= \frac{1}{n}\sum_{a=1}^{n} \pm \varphi_1 \odot_{\star} \cdots \odot_{\star} \varphi_{a-1} \odot_{\star} \mathrm{h}(\varphi_a) \odot_{\star} \varphi_{a+1} \odot_{\star} \cdots \odot_{\star} \varphi_n \, , \end{aligned} \tag{3.8}$$

or all $\varphi_1, \dots, \varphi_n \in V[2]$. We used the translation-invariance of h in (3.8) which trivializes the action of $\mathcal{R}$-matrices. Note that on generators $\varphi_a \in V[2]$, the twisted symmetric product $\odot_{\star}$ is braided graded commutative:

$$\varphi_a \odot_{\star} \varphi_b = (-1)^{|\varphi_a|\,|\varphi_b|}\, \mathrm{R}_{\alpha}(\varphi_b) \odot_{\star} \mathrm{R}^{\alpha}(\varphi_a) \, . \tag{3.9}$$

The translation invariant perturbation δ consists of the braided BV Laplacian Δ_{BV} and the braided antibracket with the interaction action $\{S_{\text{int}}, -\}_{\star}$

$$\delta = \mathrm{i}\,\hbar\Delta_{\mathrm{BV}} + \{S_{\mathrm{int}}, -\}_\star\,. \tag{3.10}$$

The braided BV Laplacian is defined by

$$\Delta_{\mathrm{BV}}(1) = 0 \quad , \qquad \Delta_{\mathrm{BV}}(\varphi_1) = 0 \quad , \qquad \Delta_{\mathrm{BV}}(\varphi_1 \odot_\star \varphi_2) = \langle \varphi_1, \varphi_2 \rangle_\star\,,$$

$$\begin{aligned}\Delta_{\mathrm{BV}}\left(\varphi_1 \odot_\star \cdots \odot_\star \varphi_n\right) = \sum_{a<b} \pm \langle \varphi_a, \mathrm{R}_{\alpha_{a+1}} \cdots \mathrm{R}_{\alpha_{b-1}}(\varphi_b)\rangle_\star \varphi_1 \odot_\star \cdots \odot_\star \varphi_{a-1} \\ \odot_\star \mathrm{R}^{\alpha_{a+1}}(\varphi_{a+1}) \odot_\star \cdots \odot_\star \mathrm{R}^{\alpha_{b-1}}(\varphi_{b-1}) \odot_\star \varphi_{b+1} \odot_\star \cdots \odot_\star \varphi_n\,,\end{aligned} \tag{3.11}$$

for all $\varphi_1, \dots, \varphi_n \in V[2]$. The action of the braided BV Laplacian encodes the braided Wick theorem, that is the quantization of the free theory. In particular, the 4-point function in free theory is given by

$$\begin{aligned}\tilde{G}_4^\star(p_1,p_2,p_3,p_4)^{(0)} &= (\mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H})^2\, \mathrm{e}^{p_1} \odot_\star \mathrm{e}^{p_2} \odot_\star \mathrm{e}^{p_3} \odot_\star \mathrm{e}^{p_4} \\ &= (\mathrm{i}\hbar)^2(2\pi)^8\big(\,\delta(p_1+p_2)\widetilde{\mathrm{G}}(p_1)\delta(p_3+p_4)\widetilde{\mathrm{G}}(p_3) \\ &\quad +\delta(p_1+p_4)\widetilde{\mathrm{G}}(p_1)\delta(p_2+p_3)\widetilde{\mathrm{G}}(p_3) \\ &\quad +\mathrm{e}^{-\mathrm{i}\,p_2\cdot\theta\,p_3}\,\delta(p_1+p_3)\widetilde{\mathrm{G}}(p_1)\delta(p_2+p_4)\widetilde{\mathrm{G}}(p_4)\ .\end{aligned} \tag{3.12}$$

The NC phase factor in the last term is the consequence of the nontrivial action of $\mathcal{R}$ -matrices in the braided Wick theorem. Unlike the standard NC QFTs (MINWALLA *et al.*, 2000; SZABO, 2003) where the Wick theorem is not deformed, in braided QFTs the free theory is deformed and the deformation is encoded in the braided Wick theorem.

The antibracket in (3.10) is the braided graded Poisson bracket

$$\{-,-\}_\star\colon \mathrm{Sym}_{\mathcal{R}}(V[2]) \otimes \mathrm{Sym}_{\mathcal{R}}(V[2]) \to \mathrm{Sym}_{\mathcal{R}}(V[2])[1]$$

defined by setting

$$\{\varphi_a, \varphi_b\}_\star = \langle \varphi_a, \varphi_b \rangle_\star = \pm\{\mathrm{R}_\alpha(\varphi_b), \mathrm{R}^\alpha(\varphi_a)\}_\star \tag{3.13}$$

for $\varphi_a \in V[2]$, and extending this to all of $\mathrm{Sym}_{\mathcal{R}}(V[2])$ as a braided graded Lie bracket which is a braided graded derivation on $\mathrm{Sym}_{\mathcal{R}}(V[2])$ in each of its slots; for example

$$\{\varphi_1, \varphi_2 \odot_\star \varphi_3\}_\star = \langle \varphi_1, \varphi_2\rangle_\star \odot_\star \varphi_3 \pm \mathrm{R}_\alpha(\varphi_2) \odot_\star \langle \mathrm{R}^\alpha(\varphi_1), \varphi_3\rangle_\star\,. \tag{3.14}$$

Interaction action For $\lambda\,\phi^4$-theory on $\mathbb{R}^{1,3}$, the interaction action $S_{\mathrm{int}} \in \mathrm{Sym}_{\mathcal{R}}(V[2])$ in (3.10) is defined by

$$S_{\mathrm{int}}^\star \equiv -\frac{1}{4!}\langle \xi, \ell_3^{\star ext}(\xi,\xi,\xi)\rangle_\star^{\mathrm{ext}}\,. \tag{3.15}$$

The contracted coordinate functions $\xi \in \mathrm{Sym}_{\mathcal{R}}(L[2]) \otimes V$ are given by

$$\xi = \int_k (\mathrm{e}_k \otimes \mathrm{e}^k + \mathrm{e}^k \otimes \mathrm{e}_k) \,, \tag{3.16}$$

where $\mathrm{e}_k(x) = \mathrm{e}^{-\mathrm{i}\, k\cdot x}$ is the basis of plane waves for V_1 with dual basis $\mathrm{e}^k(x) = \mathrm{e}^*_k(x) = \mathrm{e}^{\mathrm{i}\, k\cdot x}$ for V_2.

These bases are dual with respect to the inner product (2.6), in the sense that

$$\int_p \langle \mathrm{e}_k, \mathrm{e}^p\rangle_\star \, \mathrm{e}_p = \mathrm{e}_k \qquad \text{and} \qquad \int_k \mathrm{e}^k \, \langle \mathrm{e}_k, \mathrm{e}^p\rangle_\star = \mathrm{e}^p \,, \tag{3.17}$$

where throughout we use

$$\int \mathrm{d}^4x \; \mathrm{e}^{\pm\,\mathrm{i}\, k\cdot x} = (2\pi)^4\, \delta(k) \,. \tag{3.18}$$

The star-products among basis fields are

$$\mathrm{e}_k \star \mathrm{e}_p = \mathrm{e}^{-\frac{\mathrm{i}}{2} k\cdot\theta\, p} \; \mathrm{e}_{k+p} \,, \tag{3.19}$$

while the action of the inverse $\mathcal{R}$-matrix on them is given by

$$\mathcal{R}^{-1}(\mathrm{e}_k \otimes \mathrm{e}_p) = \mathrm{R}_\alpha(\mathrm{e}_k) \otimes \mathrm{R}^\alpha(\mathrm{e}_p) = \mathrm{e}^{\mathrm{i}\, k\cdot\theta\, p}\; \mathrm{e}_k \otimes \mathrm{e}_p \,. \tag{3.20}$$

Here we introduced the following notation $k\cdot\theta\, p = \theta^{ij} k_i p_j$. The explicit calculation of (3.15) using (2.6), (3.19) and (3.20) results in

$$S^\star_{\mathrm{int}} = \int_{k_1,k_2,k_3,k_4} V_4(k_1,k_2,k_3,k_4)\, \mathrm{e}^{k_1} \odot_\star \mathrm{e}^{k_2} \odot_\star \mathrm{e}^{k_3} \odot_\star \mathrm{e}^{k_4} \,. \tag{3.21}$$

The interaction vertex

$$V_4(k_1,k_2,k_3,k_4) = -\frac{\lambda}{4!}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{a<b} k_a\cdot\theta\, k_b}\, (2\pi)^4\, \delta(k_1+k_2+k_3+k_4) \tag{3.22}$$

coincides with the vertex of the standard noncommutative $\lambda\,\phi^4$ -theory. It has the braided symmetry

$$V_4(k_{a+1},k_a) = \mathrm{e}^{-\mathrm{i}\, k_a\cdot\theta\, k_{a+1}}\, V_4(k_1,k_2,k_3,k_4) \tag{3.23}$$

under interchange of any pair of neighbouring momenta, and also the cyclic symmetry

$$V_4(k_1,k_2,k_3,k_4) = V_4(k_4,k_1,k_2,k_3) \tag{3.24}$$

which follows from momentum conservation in (3.22).

One-loop two-point function – To illustrate both the formalism introduced above and the properties of the braided $\lambda\,\phi^4$-theory with Moyal deformation, we now present the result for the two-point function at one loop. From the definition (3.3) it follows

$$\tilde{G}_2^{\star}(p_1,p_2)^{(1)} = (\mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H})^2\,\{S_{\mathrm{int}},\mathrm{H}(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{p_2})\}_\star\,.$$

Inserting the interaction action (3.21) results in

$$\begin{aligned}\tilde{G}_2^{\star}(p_1,p_2)^{(1)} \;&= \frac{4}{2}\int_{k_1,k_2,k_3,k_4} V_4(k_1,k_2,k_3,k_4)\\ &\times\big(\langle \mathrm{e}^{k_4},\mathsf{G}(\mathrm{e}^{p_1})\rangle_\star\,(\mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H})^2\,(\mathrm{e}^{k_1}\odot_\star \mathrm{e}^{k_2}\odot_\star \mathrm{e}^{k_3}\odot_\star \mathrm{e}^{p_2})\\ &\;+\langle \mathrm{e}^{k_4},\mathsf{G}(\mathrm{e}^{p_2})\rangle_\star\,(\mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H})^2\,(\mathrm{e}^{p_1}\odot_\star \mathrm{e}^{k_1}\odot_\star \mathrm{e}^{k_2}\odot_\star \mathrm{e}^{k_3})\big)\,.\end{aligned}$$

These expressions are evaluated by using the braided Wick expansion (3.12) in momentum space. Adding all contributions we obtain the one-loop contribution to the two-point function as

$$\tilde{G}_2^{\star}(p_1,p_2)^{(1)} = \frac{\hbar^2\,\lambda}{2}\,\frac{(2\pi)^4\,\delta(p_1+p_2)}{(p_1^2-m^2)\,(p_2^2-m^2)}\int_k \frac{1}{k^2-m^2}\,. \tag{3.25}$$

This result is independent of the deformation parameter and coincides with the classical two-point function (at $\theta = 0$), including the correct sign and overall combinatorial factor. It shows that there is no UV/IR mixing in the two-point function at one-loop order, in contrast to the standard noncommutative quantum field theory (MINWALLA *et al.*, 2000; SZABO, 2003). It also suggests that there are no non-planar Feynman diagrams in perturbation theory. This appears to be a consequence of the braided symmetries of the interaction vertex due to the braided L_∞-structure, through its interplay with the braided Wick theorem. This result is also consistent with the results in (OECKL, 2000).

Braided BV quantization: λ-Minkowski spacetime

In the previous section we discussed the example of NC deformation of Minkowski spacetime (and Poincaré symmetry) in which the translations remain undeformed. This results in the standard conservation law of momenta, see (3.22). In this section we will discuss another deformation of Minkowski spacetime, but this time with broken translational invariance in the x-y plane. This deformation will result in the deformation of the momentum conservation law.

The λ-Minkowski spacetime is defined by the Drinfel'd twist (DIMITRIJEVIĆ ĆIRIĆ *et al.*, 2018a)

$$\mathcal{F} = \exp\left(-\frac{\mathrm{i}\theta}{2}\big(\partial_z\otimes\partial_\varphi - \partial_\varphi\otimes\partial_z\big)\right). \tag{4.1}$$

The commuting vector fields ∂_z and $\partial_\varphi = x\,\partial_y - y\,\partial_x$ are the generators of translations in z direction and rotations around z-axis, respectly. The pointwise product between functions $\mu(f\otimes g) = f\cdot g$ is replaced by the noncomutative $\star$-product, defined via

$$
\begin{aligned}
f \star g &= \mu \circ \mathcal{F}^{-1}(f \otimes g)\,, \\
&= f \cdot g + \frac{\mathrm{i}\theta}{2}\left(\partial_z f \cdot \partial_\varphi g - \partial_\varphi f \cdot \partial_z g\right) + \boldsymbol{O}(\theta^2)\ .
\end{aligned} \tag{4.2}
$$

The feature of noncommutativity in the space of functions is expressed via triangular $\mathcal{R}$ matrix so that the following relations hold

$$
\begin{aligned}
f \star g &= \mathcal{R}^{-1}(g \star f) = \mathrm{R}_\alpha g \star \mathrm{R}^\alpha f\,, \\
\mathcal{R}^{-1} &= \mathcal{F}^2 \equiv \mathrm{R}_\alpha \otimes \mathrm{R}^\alpha\,.
\end{aligned} \tag{4.3}
$$

The underlying braided L_∞-algebra of the $\lambda\phi^4$ scalar field theory is again the graded vector space $V = V_1 \oplus V_2$, consisting of the space of fields V_1 and the space of antifields V_2, equipped with the set of braided brackets. Out of all possible combinations from elements $\phi \in V_1$ and $\phi^+ \in V_2$, the only nontrivial action of braided brackets is when applied on fields $\phi, \phi_i \in V_1$ only and are given by

$$
\begin{aligned}
\ell_1^\star(\phi) = \ell_1(\phi) &= -(\Box + m^2)\phi\,, \\
\ell_3^\star(\phi_1, \phi_2, \phi_3) &= \lambda\phi_1 \star \phi_2 \star \phi_3\,.
\end{aligned} \tag{4.4}
$$

In order to get numbers in this formalism, we have to introduce a braided cyclic pairing of degree -3 acting nontrivial only for fields $\phi \in V_1$ and antifields $\phi^+ \in V_2$ nontrivially defined by the integral

$$
\langle \phi, \phi^+ \rangle_\star = \int d^4x\, \phi \star \phi^+ = \int d^4x\, \phi \cdot \phi^+ = \langle \phi, \phi^+ \rangle\,. \tag{4.5}
$$

Using the given L_∞-algebra data, we can write the classical action and the corresponding equations of motion as

$$
\begin{aligned}
\mathrm{S}_{\mathrm{cl}} &= \frac{1}{2!}\langle \phi, \ell_1^\star(\phi)\rangle_\star - \frac{1}{4!}\langle \phi, \ell_3^\star(\phi,\phi,\phi)\rangle_\star \\
&= \int d^4x \frac{1}{2}\phi(-\Box - m^2)\phi - \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\,,
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
\mathrm{F}_\phi &= \ell_1^\star(\phi) - \frac{1}{3!}\ell_3^\star(\phi,\phi,\phi\,) = 0 \\
0 &= (-\Box - m^2)\phi - \frac{\lambda}{3!}\,\phi \star \phi \star \phi\,.
\end{aligned} \tag{4.7}
$$

Note that formally, (4.6) has the same form as the corresponding action in the general abelian twist case (2.7).

Interaction action – To illustrate the difference between the Moyal case and the $\boldsymbol{\lambda}$-Minkowski spacetime we calculate again the one loop contribution to the two-point function.

The Feynman propagator G is the inverse of $\ell_1^\star$ given in (4.4). As in the previous section, we restrict to the trivial cohomology via $\mathrm{p} = 0$ and contracting homotopy becomes the actual propagator $h = \mathrm{G}$. In the momentum space representation it is

$$\mathrm{h}(\phi^+)(k) = \frac{\phi^+(k)}{k^2 - m^2} = \phi^+\, \widetilde{\mathrm{G}}(k)\,. \tag{4.9}$$

The interacting action functional $S^\star_{\text{int}} \in \mathrm{Sym}_{\mathcal{R}}(L[2])$ is defined as

$$S^\star_{\text{int}} \equiv -\frac{1}{4!}\langle \xi, \ell_3^{\star ext}(\xi,\xi,\xi)\rangle_\star^{\text{ext}}\,. \tag{4.10}$$

Contracted coordinate functions $\xi \in \mathrm{Sym}_{\mathcal{R}}(L[2]) \otimes V$ used in definition are constructed as combinations

$$\xi = \int_k (\mathrm{e}_k \otimes \mathrm{e}^k + \mathrm{e}^k \otimes \mathrm{e}_k) \tag{4.11}$$

and other objects are naturally extended from acting on V to acting on $\mathrm{Sym}_{\mathcal{R}}(L[2]) \otimes V$ and lending results in $\mathrm{Sym}_{\mathcal{R}}(L[2])$ but being very careful that every time one commutes any neighboring elements, one has to introduce the $\mathcal{R}$ matrix.

In the plane wave basis we have $\mathrm{e}_k(x) = \mathrm{e}^{-\mathrm{i}\,k\cdot x}$ in V_1 and $\mathrm{e}^k(x) = \mathrm{e}_k^*(x) = \mathrm{e}^{\mathrm{i}\,k\cdot x}$ in V_2[3].

These basis vectors are dual with the respect to this pairing

$$\begin{aligned} \langle \mathrm{e}_k, \mathrm{e}^p\rangle &= \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}\,k\cdot x} \star \mathrm{e}^{\mathrm{i}\,p\cdot x} \\ &= \int d^4x\, \mathrm{e}^{-\mathrm{i}\,(k+_\star \mathrm{p})\cdot x} = (2\pi)^4 \delta(p+_\star k) \\ &= (2\pi)^4\delta(p+k)\,. \end{aligned} \tag{4.12}$$

The $\star$-sum of two and three momenta is defined as

$$\begin{aligned} k+_\star p &= R\left(\frac{\theta}{2}p_z\right)k + R\left(-\frac{\theta}{2}k_z\right)p, \\ p+_\star q+_\star r &= R(r_3+q_3)p + R(-p_3+r_3)q + R(-p_3-q_3)r\,, \end{aligned} \tag{4.13}$$

with the rotation matrix R given by

$$R\left(\frac{\theta}{2}p_z\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{\theta}{2}p_z\right) & \sin\left(\frac{\theta}{2}p_z\right) & 0 \\ 0 & -\sin\left(\frac{\theta}{2}p_z\right) & \cos\left(\frac{\theta}{2}p_z\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

While the $\star$-sum does not affect t and z components (they are summed in the usual, undeformed way), it modifies the sumation of the x and y components. This reflects the

[3] We note that the plane waves are not a basis fully adapted to the twist (4.1). A more natural choice would be a symmetry adapted basis obtained by solving the wave equation in the cylindrical coordinates. A detailed discussion in the "symmetry adapted basis" will be presented in (BOGDANOVIĆ *et al.*, in preparation) .

deformed momentum conservation law in the xy-plane. The corresponding $\star$-delta functions have the following properties

$$\begin{aligned} \delta^\star(k) = &\ \int \mathrm{d}^4x\, \mathrm{e}^{\pm \mathrm{i}\, k\cdot x} = \delta(k)\,, \\ \delta^\star(k_1 +_\star k_2) = &\ \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}\, k_1\cdot x} \star \mathrm{e}^{\mathrm{i}\, k_2\cdot x} = \int d^4x\, \mathrm{e}^{-\mathrm{i}\,(k_1 +_\star k_2)\cdot x} = \delta(k_1 + k_2)\,, \\ \delta^\star(k_1 +_\star k_2 +_\star k_3) = &\ \int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}\, k_1\cdot x} \star \mathrm{e}^{\mathrm{i}\, k_2\cdot x} \star \mathrm{e}^{\mathrm{i}\, k_3\cdot x} = \delta^\star(k_2 +_\star k_3 +_\star k_1) \quad , \\ = &\ \delta^\star\big(k_1 + \ (k_2 +_\star k_3)\big) = \delta^\star\big((k_1 +_\star k_2) + k_3\big)\,. \\ \delta^\star\big((-p_2) +_\star q +_\star (-q) +_\star (-p_1)\big) = &\ \delta^\star\big((-p_2) +_\star (-p_1)\big) = \delta(p_1 + p_2)\,. \end{aligned} \tag{4.14}$$

To simplify calculations, we introduce the following notation

$$\begin{aligned} \partial_z \mathrm{e}^{k_i} = &\ i k_{iz} \mathrm{e}^{k_i}\,, \\ \partial_z \mathrm{e}_{k_j} = &\ -i k_{jz} \mathrm{e}_{k_j}\,, \\ \partial_\varphi \mathrm{e}^{k_i}(x) = &\ i\big(x k_{iy} - y k_{ix}\big)\mathrm{e}^{k_i}(x) = \Big(k_{iy}\, \partial_{k_{ix}} - k_{ix}\, \partial_{k_{iy}}\Big)\, \mathrm{e}^{k_i} = L_i \mathrm{e}^{k_i}\,, \\ \partial_\varphi \mathrm{e}_{k_j}(x) = &\ -i\big(x k_{jy} - y k_{jx}\big)\mathrm{e}_{k_j}(x) = \Big(k_{jy}\, \partial_{k_{jx}} - k_{jx}\, \partial_{k_{jy}}\Big)\, \mathrm{e}_{k_j}(x) = L_j \mathrm{e}_{k_j}(x)\,. \end{aligned}$$

The action of the $\mathcal{R}^{-1}$ matrix on the basis vectors can be written as

$$\mathrm{e}^k \star \mathrm{e}^p = \mathrm{R}_\alpha \mathrm{e}^p \star \mathrm{R}^\alpha \mathrm{e}^k = e^{\theta(p_z L_k - k_z L_p)}(\mathrm{e}^p \star \mathrm{e}^k)\,, \tag{4.15}$$

and similarly for other basis vectors.

Starting from (4.10) and applying (4.15), we calculate the interaction action

$$S^\star_{\mathrm{int}} = \int_{k_1,k_2,k_3,k_4} V_4(k_1,k_2,k_3,k_4)\, \mathrm{e}^{k_1} \odot_\star \mathrm{e}^{k_2} \odot_\star \mathrm{e}^{k_3} \odot_\star \mathrm{e}^{k_4}\,. \tag{4.16}$$

The vertex $V(k_1,k_2,k_3,k_4)$

$$V(k_1,k_2,k_3,k_4) = -\frac{\lambda}{3!} e^{\theta \sum_{a<b}(k_{bz}L_a - k_{az}L_b)} (2\pi)^4 \delta^\star\Big(k_1 +_\star k_2 +_\star k_3 +_\star k_4\Big) \tag{4.17}$$

has the following properties under the permutation of momenta

$$\begin{aligned} V(k_2,k_1,k_3,k_4) = &\ e^{\theta(k_{1z}L_2 - k_{2z}L_1)} V(k_1,k_2,k_3,k_4)\,, \\ V\big(k_1,k_3,k_2,k_{4,}\big) = &\ e^{\theta(k_{2z}L_3 - k_{3z}L_2)} V(k_1,k_2,k_3,k_4)\,, \\ V(k_2,k_3,k_1,k_4) = &\ e^{\theta(k_{1z}(L_2+L_3) - (k_{2z}+k_{3z}L_1)} V(k_1,k_2,k_3,k_4) \end{aligned} \tag{4.18}$$

and similarly for other permutations. We immediately notice that the interaction vertex (4.17) encodes the deformed conservation of momenta!

One-loop two-point function – By definition (3.3), the two-point function in momentum space is

$$\tilde{G}_2^\star(p_1,p_2) = \sum_{m=1}^{\infty} \mathrm{P}(\,\mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H} + \{S_{\mathrm{int}}, -\}_\star \mathrm{H})^m(\mathrm{e}^{p_1} \odot_\star \mathrm{e}^{p_2}))\,.$$

The first contribution is the free propagator, coming from $m = 1$ step

$$\tilde{G}_2^\star(p_1,p_2)^{(0)} = \mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H}\left(\mathrm{e}^{p_1} \odot_\star \mathrm{e}^{p_2}\right)\,. \tag{4.19}$$

This reduces to

$$\tilde{G}_2^\star(p_1,p_2)^{(0)} = (\mathrm{i}\,\hbar)(2\pi)^4\delta(p_1+p_2)\tilde{\mathrm{G}}(p_1)\,, \tag{4.20}$$

which is the commutative result. The free theory is unchanged.

The one-loop contribution to $\tilde{G}_2^\star(p_1,p_2)$ is given by

$$\tilde{G}_2^\star(p_1,p_2)^{(1)} = (\mathrm{i}\hbar\Delta_{\mathrm{BV}}\mathrm{H})^2\,\{S_{\mathrm{int}}, \mathrm{H}(\mathrm{e}^{p_1} \odot_\star \mathrm{e}^{p_2})\}_\star\,.$$

Following the same steps as in the previous section, we obtain

$$\tilde{G}_2^\star(p_1,p_2)^{(1)} = \frac{\hbar^2\,\lambda}{2}\,\frac{(2\pi)^4\,\delta(p_1+p_2)}{(p_1^2-m^2)\,(p_2^2-m^2)}\int_k \frac{1}{k^2-m^2}\,. \tag{4.21}$$

The result (4.21) is the same as (3.25) and it is identical to the result in the commutative case. Unlike in the standard NC QFT approach (DIMITRIJEVIĆ ĆIRIĆ *et al.*, 2018b), there are no non-planar diagrams, and no UV/IR mixing appears. The deformed momentum conservation law does not change the two-point function (4.21). To see nontrivial consequences of the deformed momentum conservation law, we need to calculate correlation functions with more then one vertex. This is planned for our future work.

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