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ON LOWER BOUNDS FOR THE KIRCHHOFF INDEX

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ABSTRACT. Let G be a simple graph of order $n \geq 2$ with m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ the sequence of vertex degrees and by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ the Laplacian eigenvalues of the graph G. Lower bounds for the Kirchhoff index, $Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}$, are obtained.

1 Introduction

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $E = \{e_1, e_2, \ldots, e_m\}$ be a simple connected graph of order $n \geq 3$ and size m. If vertices i and j are adjacent, we denote it as $i \sim j$. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ a sequence of vertex degrees, and by Δ and δ the greatest and the smallest vertex degrees, respectively. Let $\mathbf{A}$ be the adjacency matrix of G, and $\mathbf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ the diagonal matrix of its vertex degrees. Then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of G. Eigenvalues of $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, are the Laplacian eigenvalues of graph G.

Some well-known properties of the Laplacian eigenvalues are (see for example [3]):

$$\sum_{i=1}^{n-1}\mu_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n-1}\mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m,$$

where

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 = \sum_{i \sim j} (d_i + d_j) = \sum_{i=1}^{m} (d(e_i) + 2)$$

is the first Zagreb index introduced in [11]. In the same paper the second Zagreb index, M_2, and so called forgotten index, F, were defined as

$$M_2 = M_2(G) = \sum_{i \sim j} d_i d_j \qquad \text{and} \quad F = F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{i \sim j} (d_i^2 + d_j^2).$$

More on the invariant F one can find in [7, 9].

Matrix $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$ is the normalized Laplacian matrix of G. Its eigenvalues, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$, represent normalized Laplacian eigenvalues of G. The following is valid for ρ_i, $i = 1, 2, \ldots, n$, (see [3]):

$$\sum_{i=1}^{n-1} \rho_i = n \qquad \text{and} \qquad \sum_{i=1}^{n-1} \rho_i^2 = n + 2R_{-1},$$

where

$$R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j},$$

is the general Randić index (also called branching index) introduced in [27].

The Kirchhoff index of a connected graph is defined as (see [14]):

$$Kf(G) = \sum_{i<j} r_{ij},$$

where r_{ij} is the effective resistance distance between vertices i and j. The following more appropriate formula from application point of view was put forward in [10]

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

This, in turn, triggered the study of this invariant and its applications in various areas, including spectral graph theory, molecular chemistry, computer science, etc. (see for example [7, 9–11, 14, 18, 27]).

Before we proceed, let us define one special class of d-regular graphs Γ_d (see [25]). Let $N(i)$ be a set of all neighborhoods of the vertex i, i.e. $N(i) = \{k \mid k \in V, k \sim i\}$, and $d(i, j)$ the distance between vertices i and j. Denote by Γ_d a set of all d-regular graphs, $1 \le d \le n-1$, with diameter $D = 2$ and $|N(i) \cap N(j)| = d$. Further, denote by $t = t(G)$ a number of spanning trees of the connected graph

$$t = t(G) = \frac{1}{n}\prod_{i=1}^{n-1} \mu_i,$$

and by $ID = ID(G)$ the graph invariant called inverse degree

$$ID = ID(G) = \sum_{i=1}^{n} \frac{1}{d_i}.$$

In this paper we are concerned with the lower bounds of $Kf(G)$ which depend on some of the parameters n, m, Δ, and invariants R_{-1}, M_1, M_2 or F. Before going further, we recall some results from the literature needed for our subsequent consideration.

2 Preliminaries

In this section we outline some results for the invariants $Kf(G)$, M_1, M_2, F, t and R_{-1} that will be needed in the remainder of the paper.

In [28] the following result was proved for the $Kf(G)$:

Lemma 2.1. [28] *Let G be a simple connected graph with $n \ge 2$ vertices and m edges. Then*

$$Kf(G) \ge -1 + (n-1)\sum_{i=1}^{n} \frac{1}{d_i}, \tag{1}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{r,n-r}$, $1 \le r \le \left\lfloor \frac{n}{2} \right\rfloor$.

Remark 2.2. *We believe that equality in* (1) *holds also when $G \in \Gamma_d$ and $G \cong K_n - e$. This only increases importance of the above inequality.*

In [23] the following was proved for the general Randić index:

Lemma 2.3. [23] *Let G be a simple connected graph with $n \ge 3$ vertices and m edges. Then, for any real k with the property $\rho_1 \ge k \ge \rho_{n-1}$, holds*

$$2R_{-1} \ge \frac{n}{n-1} + \frac{n-1}{n-2}\left(k - \frac{n}{n-1}\right)^2, \tag{2}$$

with equality if and only if $k = \frac{n}{n-1}$ and $G \cong K_n$, or $k = 2$ and $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

In [13, 22, 24] for the Forgotten index the following results were established:

Lemma 2.4. [13] *Let G be a simple graph with n vertices and m edges. Then*

$$F \leq (\Delta + \delta)M_1 - 2m\Delta\delta, \tag{3}$$

with equality if and only if G is regular or bidegreed graph.

Lemma 2.5. [24] *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$F \leq 2m(\Delta^2 + \Delta\delta + \delta^2) - n\Delta\delta(\Delta + \delta), \tag{4}$$

with equality if and only if G is regular or bidegreed graph.

Lemma 2.6. [22] *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$F \leq \frac{M_1}{2m} + 2m\beta(S)(\Delta - \delta)^2, \tag{5}$$

where

$$\beta(S) = \frac{1}{2m}\sum_{i \in S} d_i \left(1 - \frac{1}{2m}\sum_{i \in S} d_i\right)$$

and S is a subset of $I = \{1, 2, \dots, n\}$which minimizes the expression

$$\left|\sum_{i \in S} d_i - m\right|.$$

Equality in (5) *holds if and only if $L(G)$ is regular.*

In [4] (see also [15]) for the first Zagreb index, M_1, the following was proved:

Lemma 2.7. [4] *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$M_1 \leq 2(\Delta + \delta)m - n\Delta\delta, \tag{6}$$

with equality if and only if G is regular or bidegreed graph.

For the same invariant in [21] the following was proved:

Lemma 2.8. [21] *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$M_1 \leq \frac{4m^2}{n} + n\alpha(n)(\Delta - \delta)^2, \tag{7}$$

where

$$\alpha(n) = \frac{1}{4}\left(1 - \frac{(-1)^{m+1}+1}{2n^2}\right).$$

Equality in (7) *holds if and only if G is regular.*

For the number of spanning trees, t, of a graph the following was proved in [5]:

Lemma 2.9. [5] *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$t \leq \frac{1}{n}\left(\frac{4m^2 - M_1 - 2m}{(n-1)(n-2)}\right)^{\frac{n-1}{2}}, \tag{8}$$

with equality if and only if $G \cong K_n$.

For the same invariant in [1] the following was proved:

Lemma 2.10. [1] *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$t \geq \left(\frac{\prod_{i=1}^{n} d_i}{2m}\right)\left(\frac{1}{n-1}\left(n^2 - (n-2)(n+2R_{-1})\right)\right)^{\frac{n-1}{2}}, \tag{9}$$

with equality if and only if $G \cong K_n$.

3 Main results

We will first prove one general result for the lower bounds of $Kf(G)$ in terms of one of the invariants R_{-1}, M_2, F or M_1.

Theorem 3.1. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq -1 + 2(n-1)R_{-1}, \tag{10}$$

$$Kf(G) \geq -1 + \frac{2(n-1)m^2}{M_2}, \tag{11}$$

$$Kf(G) \geq -1 + \frac{4(n-1)m^2}{F}, \tag{12}$$

$$Kf(G) \geq -1 + \frac{4(n-1)m^2}{\Delta M_1}. \tag{13}$$

Equalities hold if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Proof. In [19] the following inequality was proved

$$ID \geq 2R_{-1}.$$

From the inequality

$$\sum_{i\sim j} d_i d_j \sum_{i\sim j} \frac{1}{d_i d_j} \geq m^2,$$

follows that

$$R_{-1} \geq \frac{m^2}{M_2}.$$

Also, the following holds

$$2M_2 = 2\sum_{i\sim j} d_i d_j \leq \sum_{i\sim j} (d_i^2 + d_j^2) = \sum_{i=1}^{n} d_i^3 = F,$$

and

$$F_1 = \sum_{i=1}^{n} d_i^3 \leq \Delta \sum_{i=1}^{n} d_i^2 = \Delta M_1.$$

Accordingly, we have that

$$ID \geq 2R_{-1} \geq \frac{2m^2}{M_2} \geq \frac{4m^2}{F} \geq \frac{4m^2}{\Delta M_1}. \tag{14}$$

From (14) and (1) inequalities (10) – (13) are obtained. □

If in (10) – (13) invariants R_{-1}, M_2, F and M_1 are replaced with corresponding lower bounds, a number of lower bounds for $Kf(G)$ depending on various graph parameters can be obtained. In what follows we will illustrate this.

From (10) and (2) the following corollary of Theorem 3.1 is obtained.

Corollary 3.2. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any real k, $\rho_1 \geq k \geq \rho_{n-1}$, holds*

$$Kf(G) \geq n - 1 + \frac{(n-1)^2}{n-2}\left(k - \frac{n}{n-1}\right)^2, \tag{15}$$

with equality if and only if $k = \frac{n}{n-1}$ and $G \cong K_n$, or $k = 2$ and $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Since

$$\rho_1 \geq \frac{\Delta + 1}{\Delta} \geq \frac{n}{n-1} \geq \rho_{n-1},$$

according to (15), the following corollary of Theorem 3.1 holds.

Corollary 3.3. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq n - 1 + \frac{(n-1)^2}{n-2} \max\left\{ \left(\rho_1 - \frac{n}{n-1}\right)^2, \left(\rho_{n-1} - \frac{n}{n-1}\right)^2 \right\},$$

with equality if and only if $G \cong K_n$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Corollary 3.4. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq n - 1, \tag{16}$$

with equality if and only if $G \cong K_n$.

The inequality (16) was proved in [17]. It is not difficult to see that (16) can be obtained from (10) and inequality (see [16])

$$2R_{-1} \geq \frac{n}{n-1}.$$

Corollary 3.5. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq n - 1 + \frac{(n-1-\Delta)^2}{(n-2)\Delta^2},$$

with equality if and only if $G \cong K_n$.

Corollary 3.6. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{n(n-1) - \Delta}{\Delta}, \tag{17}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Proof. The inequality (17) is obtained from (10) and inequality

$$R_{-1} \geq \frac{n}{2\Delta}$$

which proved in [2]. □

The inequality (17) was proved in [25].

According to Lemma 2.4 the following corollary of Theorem 3.1 can be obtained.

Corollary 3.7. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{4(n-1)m^2}{(\Delta+\delta)M_1 - 2m\delta\Delta} - 1,$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Corollary 3.8. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{32(n-1)m^3\delta\Delta}{(\Delta+\delta)^2 M_1^2} - 1, \tag{18}$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Proof. After applying the arithmetic-geometric mean (AG) inequality on (3), i.e. on

$$F + 2m\Delta\delta \leq (\Delta+\delta)M_1,$$

the inequality

$$F \leq \frac{(\Delta+\delta)^2 M_1^2}{8m\delta\Delta}$$

is obtained. From this and (12) we obtain (18). □

From Lemma 2.5 the following corollary of Theorem 3.1 is obtained.

Corollary 3.9. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{4(n-1)m^2}{2m(\Delta^2+\Delta\delta+\delta^2) - n\Delta\delta(\Delta+\delta)} - 1,$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Similarly, from Lemma 5 and (12) the following corollary of Theorem 3.1 is obtained.

Corollary 3.10. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{8(n-1)m^3}{M_1^2 + 4m^2\beta(S)(\Delta-\delta)^2} - 1,$$

where

$$\beta(S) = \frac{1}{2m}\sum_{i\in S} d_i \left(1 - \frac{1}{2m}\sum_{i\in S} d_i\right)$$

and S is a subset of $I = \{1, 2, \ldots, n\}$ which minimizes the expression

$$\left| \sum_{i \in S} d_i - m \right| .$$

Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Corollary 3.11. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{4(n-1)m^2}{\Delta(2m(\Delta+\delta) - n\Delta\delta)} - 1,$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Proof. The required inequality is obtained from (6) and (13). □

Corollary 3.12. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{4n(n-1)\delta}{(\Delta+\delta)^2} - 1,$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Proof. After applying the AG inequality on (6), i.e. on

$$M_1 + n\Delta\delta \leq 2m(\Delta+\delta),$$

the inequality

$$M_1 \leq \frac{(\Delta+\delta)^2 m^2}{n\Delta\delta}$$

is obtained (see [6, 8, 12, 20]). The required inequality is obtained from the above inequality and (13). □

From (7) and (13) the following corollary of Theorem 3.1 is obtained.

Corollary 3.13. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{4n(n-1)m^2}{\Delta(4m^2 + n^2\alpha(n)(\Delta-\delta)^2)} - 1,$$

where

$$\alpha(n) = \frac{1}{4}\left(1 - \frac{(-1)^{n+1}+1}{2n^2}\right).$$

Equality holds if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Corollary 3.14. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$Kf(G) \geq \frac{2m(n-1)}{\Delta^2} - 1,$$

with equality if and only if $G \cong K_n$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $G \in \Gamma_d$.

Proof. The required result is obtained from (13) and inequality

$$M_1 \leq 2m\Delta.$$

□

The following corollary of Theorem 3.1 sets up a lower bound for $Kf(G)$ in terms of parameters n and m and the invariant t.

Corollary 3.15. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \geq \frac{4(n-1)m^2}{\Delta(4m^2 - 2m - (n-1)(n-2)(nt)^{\frac{2}{n-1}})} - 1, \tag{19}$$

with equality if and only if $G \cong K_n$.

Proof. From inequality (8) follows

$$M_1 \leq 4m^2 - 2m - (n-1)(n-2)(nt)^{\frac{2}{n-1}}.$$

From the above and inequality (13) we arrive at (19). □

Similarly, the following can be proved:

Corollary 3.16. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges, and let t be the total number of spanning trees of G. Then*

$$Kf(G) \geq \frac{n-1}{n-2}\left(2n - (n-1)\left(\frac{2mt}{\prod_{i=1}^{n} d_i}\right)^{\frac{2}{n-1}}\right) - 1,$$

with equality if and only if $G \cong K_n$.

Proof. From (9) follows

$$2R_{-1} \geq \frac{1}{n-2}\left(2n - (n-1)\left(\frac{2mt}{\prod_{i=1}^{n} d_i}\right)^{\frac{2}{n-1}}\right).$$

From the above and inequality (10) we obtain the required result. □

Let us note that the connectivity condition for the graph G does not deteriorate the generality of the results. Namely, a graph G can be observed as a union of connected components as well.

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