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ELASTIC-PLASTIC STRESSES IN A THIN ROTATING DISK WITH SHAFTHAVING DENSITY VARIATION PARAMETER UNDER STEADY-STATE TEMPERATURE

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ABSTRACT. Steady thermal stresses in a rotating disc with shaft having density variation parameter subjected to thermal load have been derived by using Seth's transition theory. Neither the yields criterion nor the associated flow rule is assumed here. Results are depicted graphically. It has been seen that compressible material required higher percentage increased angular speed to become fully-plastic as compare to rotating disc made of incompressible material. Circumferential stresses are maximal at the outer surface of the rotating disc. With the introduction of thermal effect it decreases the value of radial and circumferential stresses at inner and outer surface for fully-plastic state.

Key words: Stresses, displacement, rotating disc, angular speed, shaft, temperature, density.

INTRODUCTION

Disc plays an important role in machine design. Stress analysis of rotating discs has an important role in engineering design. Rotating discs are the most critical part of rotors, turbines motor, compressors, high speed gears, flywheel, sink fits, turbo jet engines and computer's disc drive etc.The problem of thin rotating flat discs made of isotropic material has been studied extensively [1-3]. CHAKRABARTY [1] and HEYMAN [2] solved the problem for the plastic state by utilizing the solution in the elastic state and consider the plastic range with the help of Tresca's yield condition. Further, to obtain the elastic-plastic stresses, these authors matched the elastic and plastic stresses at the same radius $r = c$ of the disc. Perfectly elasticity and ideal plasticity are two extreme properties of the material and the use of ad-hoc rule like yield condition amounts to divide the two extreme properties by a sharp line, which is not physically possible. Seth's transition theory[4] does not required any assumptions like an yield criterion, incompressibility condition, associated flow rule and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory [4] utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems [4-19].SETH [5]has defined the generalized principal strain measure as:

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1-2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ii} = \frac{1}{n}\left[1-\left(1-2\overset{A}{e}_{ii}\right)\right], \ (i=1,2,3) \qquad (1)$$

where n is the measure and $\overset{A}{e}_{ii}$ is the almansi finite strain components. For n =-2, -1,0, 1, 2it gives Cauchy, Green Hencky, Swainger and Almansi measures, respectively.

In this research paper we discusselastic-plastic transitional stresses in a thin rotating disk with shaft having density variation parameter under steady state temperature by using Seth's transition theory. The density of disc is assumed to vary along the radius in the form:

$$\rho = \rho_0 \left(r / b\right)^{-m} \qquad (2)$$

where ρ_0 is the constant density at $r = b$ and $m$is the density variation parameter. Result obtained have been numerically and depicted graphically.

MATHEMATICAL MODEL

Consider a thin disc of isotropic and homogeneous material having variable density with central bore of inner radius a and external radius b. The annular disc is mounted on a rigid shaft. The disc is rotating with angular speed ω of gradually increasing magnitude about an axis perpendicular to its plane and passed through the center as shown in Fig. 1. The thickness of disc is assumed to be constant and sufficiently small so that it is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero. We assume that steady state temperature Θ_0 is applied on the internal surface of the disc.

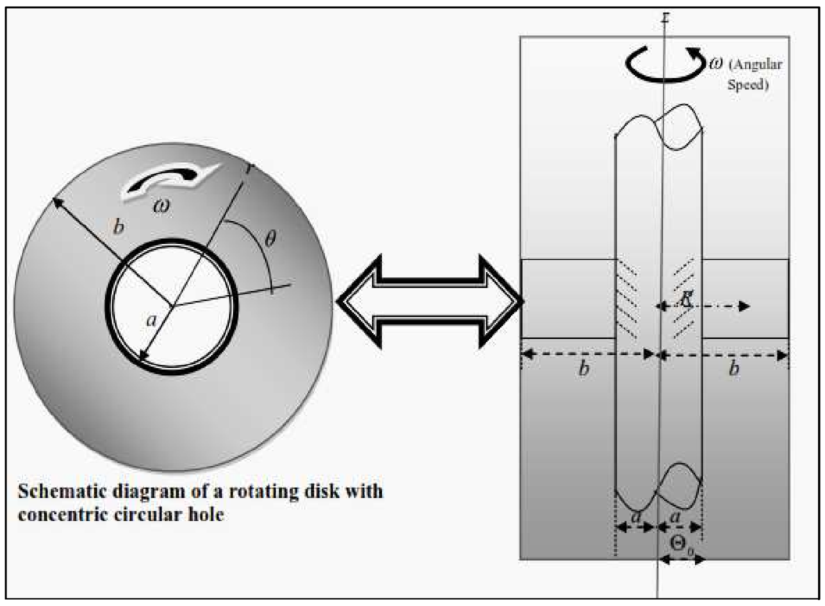

Figure 1. Geometry of Rotating Disc.

Boundary conditions

The disk considered in the present study having variable density and subjected to a thermal load. The inner surface of the disk is assumed to be fixed to a shaft. The outer surface of the disk is free from mechanical load. Thus, the boundary conditions of the problem are given by:

(i) $r = a \ u = 0$

(ii) $r = b, \ T_{rr} = 0$ (3)

where u and T_{rr} denote displacement and stress along the radial direction.

Formulation of the Problem

Displacement components in cylindrical polar coordinates (r, θ, z), as:

$$u = r(1-\beta), v = 0, w = dz \tag{4}$$

where β is function of $r = \sqrt{x^2 + y^2}$ only and *d* is a constant.

The finite strain components are given by Seth [5]:

$$\overset{A}{e}_{rr} \equiv \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-(r\beta'+\beta)^2\right], \overset{A}{e}_{\theta\theta} \equiv \frac{u}{r} - \frac{u^2}{2r^2} = \frac{1}{2}\left[1-\beta^2\right]$$

$$\overset{A}{e}_{zz} \equiv \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\left[1-(1-d)^2\right], \overset{A}{e}_{r\theta} = \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0 \tag{5}$$

where $\beta' = d\beta/dr$ and meaning of superscripts "^{A}" is Almansi.

Substituting eq.(5)in eq. (1), the generalized components of strain are given by:

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right], e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right], e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right], e_{r\theta} = e_{\theta z} = e_{zr} = 0 \tag{6}$$

where $\beta' = d\beta/dr$.

The stress –strain relations for thermo elastic isotropic material are given by [20]:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij} - \xi\Theta\delta_{ij}, \ (i, j = 1,2,3) \tag{7}$$

where T_{ij} is the stress components, λ and μ are Lame's constants and $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta and $\xi = \alpha(3\lambda + 2\mu), \alpha$ being the coefficient of thermal expansion and Θ is the rise of temperature. Further, Θ has to satisfy

$$\nabla^2\Theta = 0 \tag{8}$$

Eq. (7) for this problem becomes:

$$T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr} + e_{\theta\theta}\right] + 2\mu e_{rr} - \frac{2\mu\xi\Theta}{(\lambda+2\mu)}, T_{\theta\theta} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr} + e_{\theta\theta}\right] + 2e_{\theta\theta} - \frac{2\mu\xi\Theta}{(\lambda+2\mu)},$$

$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0 \tag{9}$$

Substituting eq. (5) in eq. (6), the strain components in terms of stresses are obtained as [6]:

$$e_{rr} = \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-(r\beta'+\beta)^2\right] = \frac{1}{E}\left[T_{rr} - \left(\frac{1-C}{2-C}\right)T_{\theta\theta}\right] + \alpha\Theta,$$

$$e_{\theta\theta} = \frac{u}{r} - \frac{u^2}{2r^2} = \frac{1}{2}\left[1-\beta^2\right] = \frac{1}{E}\left[T_{\theta\theta} - \left(\frac{1-C}{2-C}\right)T_{rr}\right] + \alpha\Theta,$$

$$e_{zz} = \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\left[1-(1-d)^2\right] = -\frac{(1-C)}{E(2-C)}\left[T_{rr} - T_{\theta\theta}\right] + \alpha\Theta,$$

$$e_{r\theta} = e_{\theta z} = e_{zr} = 0. \tag{10}$$

where *E* is the Young's modulus and *C* is compressibility factor of the material in term of Lame's constant, there are given by $E = \mu(3\lambda+2\mu)/(\lambda+\mu)$ and $C = 2\mu/(\lambda+2\mu)$.

Substituting eq. (6) in eq. (9), one get

$$T_{rr} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{1-C+(2-C)(P+1)^n + \frac{nC\xi\Theta}{2\mu\beta^n}\right\}\right],$$

$$T_{\theta\theta} = \frac{2\mu}{n}\left[3-2C-\beta^n\left\{2-C+(1-C)(P+1)^n + \frac{nC\xi\Theta}{2\mu\beta^n}\right\}\right],$$

$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0. \tag{11}$$

where $r\beta' = \beta P$.

The equations of equilibrium are all satisfied except:

$$\frac{d}{dr}\left(rT_{rr}\right)-T_{\theta\theta}+\rho\omega^2 r^2=0 \tag{12}$$

where ρ variable density of the material of the rotating disc.
The temperature field satisfying Laplace eq. (8) with boundary condition
$\Theta=\Theta_0$ at $r = a$
$\Theta=0$ at $r = b$.
where Θ_0 is constant, given by:

$$\Theta=\Theta_0\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)} \tag{13}$$

Using eqs. (11), (12) and (13), one gets a non- linear differential equation in β as:

$$\left(2-C\right)n\beta^{n+1}P\left(P+1\right)^{n-1}\frac{dP}{d\beta}=\left[\frac{n\rho\omega^2r^2}{2\mu}+\beta^n\begin{bmatrix}1-\left(P+1\right)^n\\ -nP\left\{1-C+\left(2-C\right)\left(P+1\right)^n\right\}\end{bmatrix}-\frac{nC\xi\bar{\Theta}_0}{2\mu}\right] \tag{14}$$

where $\bar{\Theta}_0=\Theta_0/\ln\left(a/b\right)$ and $r\beta'=\beta P$ (P is function of β and β is function of r) and $\beta'=d\beta/dr$ (P is function of β and β is function of r only).

Solution through the Problem

For finding the plastic stress, the transition function is taken through the principal stress (see SETH's [4, 5] and PANKAJ THAKUR [6-21]) at the transition point $P\to\pm\infty$. The transition function τ is defined as:

$$\tau=\frac{n}{2\mu}\left[T_{\theta\theta}-C\xi\Theta\right]=\left[\left(3-2C\right)-\beta^n\left\{2-C+\left(1-C\right)\left(P+1\right)^n\right\}-\frac{nC\xi\Theta}{\mu}\right] \tag{15}$$

Taking the logarithmic differentiation of eq. (15) with respect to r, one get:

$$\frac{d\left(\log\tau\right)}{dr}=-\left(\frac{n\beta^nP}{r}\right)\left[\frac{2-C+\left(1-C\right)\left(P+1\right)^{n-1}\left\{\left(P+1\right)+\beta\frac{dP}{d\beta}\right\}}{\left(3-2C\right)-\beta^n\left\{2-C+\left(1-C\right)\left(P+1\right)^n\right\}-\frac{nC\xi\Theta}{\mu}}\right] \tag{16}$$

Substituting the value $dP/d\beta$ from eq. (14) in eq. (16) and taking the asymptotic value $P\to\pm\infty$ and integrating, one get:

$$\tau = Ar^{\nu-1} \tag{17}$$

where $\nu=1-C/2-C$ and A is a constant of integration can be determined by boundary conditions.
Eqs. (15) and (17) gives:

$$T_{\theta\theta}=\left(\frac{2\mu}{n}\right)Ar^{\nu-1}+\frac{C\xi\Theta_0\ln\left(r/b\right)}{\ln\left(a/b\right)} \tag{18}$$

Substituting eq. (18) in eq. (12) then using eq. (2) and integrating, one get:

$$T_{rr}=\left(\frac{2\mu}{n\nu}\right)Ar^{\nu-1}-\frac{\rho_0\omega^2b^mr^{2-m}}{\left(3-k\right)}+\frac{B}{r}+\frac{C\xi\Theta_0\ln\left(r/b\right)}{\ln\left(a/b\right)}-\frac{C\xi\Theta_0}{\ln\left(a/b\right)} \tag{19}$$

where B is a constant of integrationcan be determined by boundary conditions.

Substituting eqs. (18) and (19) in second equation of eq. (10), one get:

$$\beta = \sqrt{1-\frac{2\nu}{E}\left[\frac{\rho_0\omega^2 b^m r^{2-m}}{3-m} - \frac{B_1}{r} + \frac{\alpha E\,\Theta_0\left(2-C\right)}{\ln\left(a/b\right)} + \frac{2\left(2-C\right)\alpha E\,\Theta_0 \ln\left(r/b\right)}{\ln\left(a/b\right)}\right]} \tag{20}$$

where $C\xi = \alpha E\left(2-C\right)$.

Substituting eq. (20) in eq. (4), one get:

$$u = r - r\sqrt{1-\frac{2\nu}{E}\left[\frac{\rho_0\omega^2 b^m r^{2-m}}{3-m} - \frac{B_1}{r} + \frac{\alpha E\,\Theta_0\left(2-C\right)}{\ln\left(a/b\right)} + \frac{2\left(2-C\right)\alpha E\,\Theta_0 \ln\left(r/b\right)}{\ln\left(a/b\right)}\right]} \tag{21}$$

where *E*= $2\mu\left(3-2C\right)/\left(2-C\right)$ is the Young's modulus in term of compressibility factor can be expressed as.

Using boundary condition (3) and (13) in eqs. (19) and (21), one get:

$$A = \frac{\rho_0\omega^2 b^m n\nu\left(b^{3-m}-a^{3-m}\right)}{2\mu\left(3-m\right)b^{\nu}} + \frac{\alpha E\,\Theta_0 n\left(1-C\right)\left(b-a\right)}{2\mu\ln\left(a/b\right)b^{1-C/2-C}} - \frac{\alpha E\,\Theta_0 na}{\mu b^{1-C/2-C}} \tag{22}$$

$$B = \frac{\rho_0\omega^2 b^m a^{3-m}}{\left(3-m\right)} + \frac{\alpha E\,\Theta_0\left(2-C\right)a}{\ln\left(a/b\right)} + \frac{2\left(2-C\right)\alpha E\,\Theta_0 a}{\left(1-C\right)} \tag{23}$$

Substituting eqs. (22) and (23) in eqs. (18), (19), and (21) respectively, one get the transitional stresses and displacement as:

$$T_{\theta\theta} = \left(\frac{\rho_0\omega^2\nu b^m\begin{pmatrix}b^{3-m}\\-a^{3-m}\end{pmatrix}}{\left(3-m\right)r}\left(\frac{r}{b}\right)^{\nu} + \alpha E\,\Theta_0\left(2-C\right)\begin{bmatrix}\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)} - \frac{2a}{\left(2-C\right)r}\left(\frac{r}{b}\right)^{1-C/2-C}\\ + \frac{\left(1-C\right)\left(b-a\right)}{r\left(2-C\right)\ln\left(a/b\right)}\left(\frac{r}{b}\right)^{1-C/2-C}\end{bmatrix}\right) \quad \forall \quad m \neq 3 \tag{24}$$

$$T_{rr} = \left(\begin{array}{l}\frac{\rho_0\omega^2 b^m}{\left(3-m\right)r}\begin{bmatrix}\left(b^{3-m}-a^{3-m}\right)\left(\frac{r}{b}\right)^{\nu}\\ -r^{3-m}+a^{3-m}\end{bmatrix} + \frac{\alpha E\,\Theta_0\left(2-C\right)}{\ln\left(a/b\right)}\begin{bmatrix}\ln\left(r/b\right)+\frac{a}{r}-1+\\ \frac{\left(b-a\right)}{r}\left(\frac{r}{b}\right)^{1-C/2-C}\end{bmatrix}\\ + \frac{2\alpha E\,\Theta_0\left(2-C\right)}{\left(1-C\right)}\left[\frac{a}{r}-\frac{a}{r}\left(\frac{r}{b}\right)^{1-C/2-C}\right]\end{array}\right) \quad \forall \quad m \neq 3 \tag{25}$$

$$u = r - r\sqrt{1-\frac{2\nu}{E}\left\{\begin{array}{l}\left(\frac{\rho_0\omega^2 b^m}{\left(3-m\right)r}\left[r^{3-m}-a^{3-m}\right]\right) + \frac{\alpha E\,\Theta_0\left(2-C\right)\left(r-a\right)}{r\ln\left(a/b\right)}\\ + \frac{2\left(2-C\right)\alpha E\,\Theta_0}{\left(1-C\right)}\left[\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)}-\frac{a}{r}\right]\end{array}\right\}} \quad \forall \quad m \neq 3 \tag{26}$$

$$\text{and}\, T_{rr} - T_{\theta\theta} = \left| \begin{array}{l} \dfrac{\rho_0 \omega^2 b^2}{(3-m)R}\left[(1-\nu)\left(1-R_0^{\;3-m}\right)R^{\nu} - R^{3-m} + R_0^{\;3-m}\right] \\ +\alpha E\,\Theta_0 \left[\begin{array}{l} \dfrac{2R_0}{R\nu} - \dfrac{2R_0}{(1-C)} R^{\nu-1} + (1-R_0) R^{\nu-1} \\ + \dfrac{(2-C)}{\ln R_0}\left(\dfrac{R_0 - R}{R}\right) \end{array}\right] \end{array} \right| \quad \forall \quad m \neq 3 \qquad (27)$$

where $R_0 = a/b$ and $R = r/b$ in non dimensional form.

Initial Yielding: The maximum value $\left|T_{rr} - T_{\theta\theta}\right|$ occurs at the radius $R = R_1$ (say), which depends upon the value of *m* and *C*. For example if we take *C* = 0, 0.25, 0.5 yielding starts at the internal surface for *m* = -1.9, -1.6, -1.2 respectively and for values *m* = -5.1, -4.9, -3.9 yielding starts at the mid surface. For the values $m = -3.1\times10^9$, -2.9×10^9 , -1.6×10^9; $\left|T_{rr} - T_{\theta\theta}\right|$ become neither maximum nor minimum values at the external surface $R_1 = 1$ *i.e.* yielding does not occurs at the external surface. For yielding at $R = R_1$, eq. (27) becomes:

$$\left|T_{rr} - T_{\theta\theta}\right|_{R=R_1} = \left| \begin{array}{l} \dfrac{\rho_0 \omega^2 b^2}{(3-m)R_1}\left[(1-\nu)\left(1-R_0^{\;3-m}\right)R_1^{\nu} - R_1^{3-m} + R_0^{\;3-m}\right] \\ +\alpha E\,\Theta_0 \left[\dfrac{2R_0}{R_1\nu} - \dfrac{2R_0}{(1-C)} R_1^{\nu-1} + (1-R_0) R_1^{\nu-1} + \dfrac{(2-C)}{\ln R_0}\left(\dfrac{R_0 - R_1}{R_1}\right)\right] \end{array} \right| \equiv Y(\textit{yielding say})$$

where *Y* is the yielding stress.
The angular speed necessary for initial yielding is given by:

$$\Omega_i^2 = \frac{\rho_0 \omega_i^2 b^2}{Y} = \left(\left|R_1 S\right| - \left| \left(\frac{\alpha E\,\Theta_0}{Y}\right) S \left[\frac{2R_0}{\nu} - \frac{2R_0}{(1-C)} R_1^{\nu} + (1-R_0) R_1^{\nu} + \frac{(2-C)}{\ln R_0} \begin{pmatrix} R_0 \\ -R_1 \end{pmatrix} \right] \right| \right) \qquad (28)$$

where $S = \dfrac{(3-m)R_1}{\left[(1-\nu)\left(1-R_0^{\;3-m}\right)R_1^{\nu} - R_1^{3-m} + R_0^{\;3-m}\right]}$ and $\omega_i = \dfrac{\Omega_i}{b}\sqrt{\dfrac{Y}{\rho_0}}$.

We introduce the following non-dimensional components:

$R = r/b$, $R_0 = a/b$, $\sigma_r = T_{rr}/Y$, $\sigma_\theta = T_{\theta\theta}/Y$, $\bar{u} = u/b$, $\Theta_1 = \alpha E\,\Theta_0 / Y$, $\Omega^2 = \rho_0 \omega^2 b^2 / Y$ and $H = Y/E$.

Elastic-plastic transitional stresses, angular speed and displacement from equations (24), (25), (28) and (26) in non-dimensional form become:

$$\sigma_\theta = \left(\frac{\Omega_i^2 \nu \left(1 - R_0^{3-m}\right) R^{\nu-1}}{(3-m)} \right) + \Theta_1 (2-C) \left[\frac{(1-R_0)(1-C)}{(2-C)\ln R_0} R^{\nu-1} + \frac{\ln R}{\ln R_0} - \frac{2R_0}{(2-C)} R^{\nu-1} \right] \qquad (29)$$

$$\sigma_r = \left[\frac{\Omega_i^2}{R(3-m)} \left[\begin{array}{l} \left(1-R_0^{3-m}\right) R^{\nu} \\ -R^{3-m} + R_0^{3-m} \end{array} \right] + \frac{\Theta_1 (2-C)}{\ln R_0} \left[\begin{array}{l} \ln R + \dfrac{R_0}{R} - 1 \\ +(1-R_0) R^{\nu-1} \end{array} \right] + \frac{2\Theta_1}{\nu} \left[\frac{R_0}{R} \left(1 - R^{\nu}\right) \right] \right] \qquad (30)$$

$$\Omega_i^2 = \left(|R_1 S| - \left| \Theta_1 S \left[\begin{array}{l} \frac{2R_0}{\nu} - \frac{2R_0}{(1-C)} R_1^{\nu} + (1-R_0) R_1^{\nu} \\ + \frac{(2-C)}{\ln R_0}(R_0 - R_1) \end{array} \right] \right| \right) \tag{31}$$

$$\text{and } U = R - R\sqrt{1 - 2\nu H \left\{ \begin{array}{l} \frac{\Omega_i^2}{R(3-m)}\left[R^{3-m} - R_0^{3-m}\right] + \frac{\Theta_1(2-C)(R-R_0)}{R \ln R_1} \\ + \frac{2(2-C)\Theta_1}{(1-C)}\left[\frac{\ln R}{\ln R_0} - \frac{R_0}{R}\right] \end{array} \right\}} \tag{32}$$

Fully Plastic State: Stresses and displacement at the inner boundary satisfied the inequality $T_{rr} > T_{\theta\theta} > T_{zz}(=0)$ and yielding occurs at the inner radius. For fully-plastic state $C \to 0 \; i.e. \; \nu = 1/2$. Two plastic zones for fully plastic state were considered as shown in Fig. 2. There are two plastic zones:

(i) **Inner-plastic zone:**
$T_{rr} > T_{\theta\theta} > T_{zz}(=0) \,; a \le r \le r_1$
or $\sigma_r > \sigma_\theta > \sigma_z(=0) \,; R_0 \le R \le R_1$.

(ii) **Outer-plastic zone:**
$T_{\theta\theta} > T_{rr} > T_{zz}(=0) \,, r_1 \le r \le b$
or $\sigma_\theta > \sigma_r > \sigma_z(=0) \;\; R_1 \le R \le 1$.

Where r_1 is the radius of inner plastic zone.

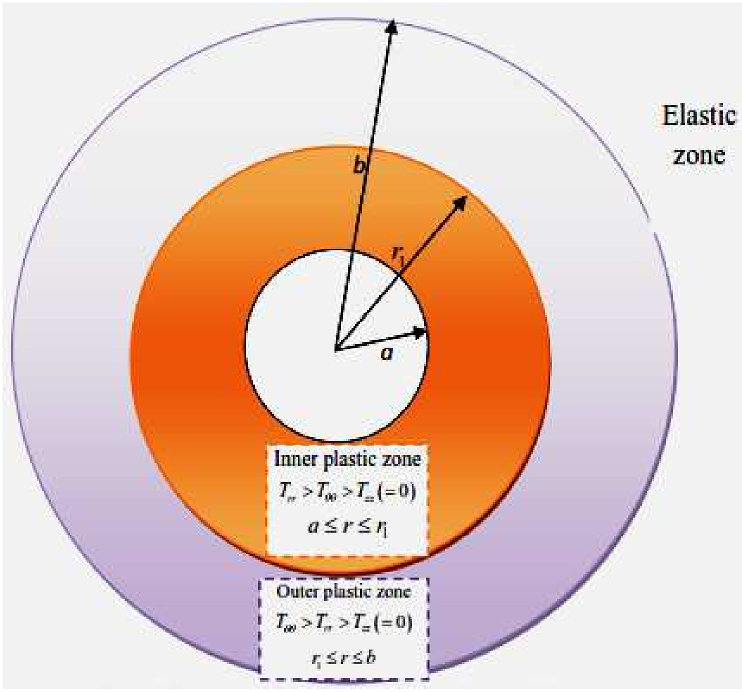

Figure 2. Two different plastic zones around the disc for fully plastic state.

For *Inner-plastic zone*, eq. (27) becomes:

$$|T_{rr} - T_{\theta\theta}|_{R=R_0} = \left| \frac{\rho_0 \omega_{f^*}^2 b^2 \left(1 - R_0^{3-m}\right)}{2(3-m)\sqrt{R_0}} + \frac{\alpha E \Theta_0}{\sqrt{R_0}}\left[4\sqrt{R_0} - 3R_0 + 1\right] \right| \equiv Y^*(say)$$

and the angular speed required for fully plastic state is given by:

$$\Omega_{f^*}^2 = \frac{\rho_0 \omega_{f^*}^2 b^2}{Y^*} = \left[\left| \frac{2(3-m)\sqrt{R_0}}{\left(1-R_0^{3-m}\right)} \right| - \left| \Theta_1 \left(\frac{2(3-m)}{\left(1-R_0^{3-m}\right)} \right) \left[4\sqrt{R_0} - 3R_0 + 1\right] \right| \right] \tag{33}$$

where $\omega_{f^*} = \frac{\Omega_{f^*}}{b}\sqrt{\frac{Y}{\rho_0}}$ and $\frac{\alpha E \Theta_0}{Y^*} = \Theta_1$.

Using equation (33) in eqs. (29), (30), (32) by taking $C \to 0 \; i.e. \; \nu = 1/2$, we get the stresses and displacement for the *inner plastic zone* as:

$$\sigma_\theta^* = \left(\frac{\Omega_{f^*}^2 \left(1 - R_0^{3-m}\right)}{2(3-m)\sqrt{R}} \right) + 2\Theta_1 \left[\frac{(1-R_0)}{2\sqrt{R}\ln R_0} + \frac{\ln R}{\ln R_0} - \frac{R_0}{\sqrt{R}} \right] \tag{34}$$

$$\sigma_r^* = \left[\frac{\Omega_{f^*}^2}{R(3-m)}\left[\begin{array}{l}(1-R_0^{3-m})\sqrt{R}\\ -R^{3-m}+R_0^{3-m}\end{array}\right] + \frac{2\Theta_1}{\ln R_0}\left[\ln R + \frac{R_0}{R} - 1 + \frac{(1-R_0)}{\sqrt{R}}\right] + \Theta_1\left[\frac{R_0}{R}\left(1-\sqrt{R}\right)\right]\right] \tag{35}$$

and $U_{f^*} = R - R\sqrt{1 - H\left\{\frac{\Omega_{f^*}^2}{R(3-m)}\left[R^{3-m} - R_0^{3-m}\right] + \frac{2\Theta_1(R-R_0)}{R\ln(R_0)} + 4\Theta_1\left[\frac{\ln R}{\ln R_0} - \frac{R_0}{R}\right]\right\}}$ (36)

For *outer plastic zone* eq. (27) becomes:

$$\left|T_{rr} - T_{\theta\theta}\right|_{R=1} = \left|-\frac{\rho_0\omega_{f^{**}}^2 b^2\left(1-R_0^{3-m}\right)}{2(3-m)} + \alpha E\,\Theta_0\left[R_0 + 1 + \frac{2}{\ln R_0}(R_0 - 1)\right]\right| = Y^{**}\,(say)$$

and the and the required angular speed is given by:

$$\Omega_{f^{**}}^2 = \frac{\rho_0\omega_{f^{**}}^2 b^2}{Y^{**}} = \left|\frac{2(3-m)}{1-R_0^{3-m}}\right| - \left|\Theta_1\left(\frac{2(3-m)}{1-R_0^{3-m}}\right)\left[R_0 + 1 + \frac{2}{\ln R_0}(R_0 - 1)\right]\right| \tag{37}$$

where $\omega_{f^{**}} = \frac{\Omega_{f^{**}}}{b}\sqrt{\frac{Y}{\rho_0}}$ and $\frac{\alpha E\,\Theta_0}{Y^{**}} = \Theta_1$.

Using equation (37) in eqs. (29), (30), (32) by taking $C \to 0\, i.e.\, \nu = 1/2$, one get the stresses and displacement for the *outer plastic zone* as:

$$\sigma_\theta^{**} = \left(\frac{\Omega_{f^{**}}^2\left(1-R_0^{3-m}\right)}{2(3-m)\sqrt{R}}\right) + 2\Theta_1\left[\frac{(1-R_0)}{2\sqrt{R}\ln R_0} + \frac{\ln R}{\ln R_0} - \frac{R_0}{\sqrt{R}}\right] \tag{38}$$

$$\sigma_r^{**} = \left[\frac{\Omega_{f^{**}}^2}{R(3-m)}\left[\begin{array}{l}(1-R_0^{3-m})\sqrt{R}\\ -R^{3-m}+R_0^{3-m}\end{array}\right] + \frac{2\Theta_1}{\ln R_0}\left[\ln R + \frac{R_0}{R} - 1 + \frac{(1-R_0)}{\sqrt{R}}\right] + \Theta_1\left[\frac{R_0}{R}\left(1-\sqrt{R}\right)\right]\right] \tag{39}$$

and $U_{f^{**}} = R - R\sqrt{1 - H\left\{\frac{\Omega_{f^{**}}^2}{R(3-m)}\left[R^{3-m} - R_0^{3-m}\right] + \frac{2\Theta_1(R-R_0)}{R\ln R_0} + 4\Theta_1\left[\frac{\ln R}{\ln R_0} - \frac{R_0}{R}\right]\right\}}$ (40)

RESULTS AND DISCUSSION

For calculating the stresses, angular speed and displacement based on the above analysis, the following values have been taken: C = 0.00, 0.25, 0.5 and 0.75, $\Theta_0 = 700^0 F$ and $\alpha = 5.0 \times 10^{-5}\deg F^{-1}$ for Methyl Methacrylate [21], $\Theta_1 = 0, 0.0175$ and 0.07 respectively.

In Tab. 1, angular speed required for initial yielding Ω_i^2 and fully-plastic state Ω_f^2 in a rotating disc having variable density for different values of *m*, *C* and Θ_1 has been given. It can be seen from the Tab. 1 that yielding occurs at any radius $R = R_1$ or at the internal surface $R_1 = 0.5$ or at the mid surface $R_1 = 0.7$ of the disc depending upon the values of *m* and *C*. For example yielding occurs at the internal surface of the disc made of compressible material (*C* = 0.25) at a angular speed 2.82511676 for *m* = -1.6 whereas yielding occurs at the middle surface at the angular speed 7.72178021 for *m* = -3. It is also seen from tab.1 that rotating disc having variable density and made of incompressible material yields at a higher angular speed as compare to disc made of compressible material. Compressible material of rotating disc with shaft having variable density required higher percentage increased angular speed to become fully-plastic as compare to incompressible material. In Tab. 2, angular speed required for initial yielding Ω_i^2 and fully-plastic state Ω_f^2 of a rotating disc for different values of *m*=0, 2.9 and $\Theta_1 = 0.00, 0.0175, 0.07$ has been given. It is seen that with the effect of temperature, rotating disc required higher percentage angular speed to become fully plastic state with increase temperature for *m*=2.9 but reverse in case *m*=0.

Table 1. Angular speed for initial yielding Ω_i^2 and fully plastic state Ω_f^2 of a rotating disc for different values of m, C and Θ_1 .

	Temperature	Compressibility	Density variation	Yielding Occurs at	Rotating Disc having Variable density $\rho=\rho_0(r/b)^{-m}$		Angular speed required for initial yielding	Angular speed required for fully -plastic state	Percentage increase in Angular speed %
	Θ_1	C	m	R	$r=a$	$r=b$	Ω_i^2	Ω_f^2	$\left(\sqrt{\Omega_f^2/\Omega_i^2}-1\right)\times 100$
	0	0	-1.9		$\rho=\rho_0(0.267943)$	$\rho=\rho_0$	3.58489172	10.13960499	68.1792831 %
	0.0175	0	-1.9		$\rho=\rho_0(0.267943)$	$\rho=\rho_0$	3.37831007	9.617444093	68.7252503 %
	0.07	0	-1.9		$\rho=\rho_0(0.267943)$	$\rho=\rho_0$	2.75856513	8.050961414	70.8371241 %
	0	0.25	-1.6		$\rho=\rho_0(0.329877)$	$\rho=\rho_0$	2.82511676	9.595673992	84.2975414 %
	0.0175	0.25	-1.6		$\rho=\rho_0(0.329877)$	$\rho=\rho_0$	2.65562129	9.101524002	85.128801 %
	0.07	0.25	-1.6	$R_1=0.5$	$\rho=\rho_0(0.329877)$	$\rho=\rho_0$	2.14713488	7.619074033	88.3742052 %
	0	0.5	-1.2		$\rho=\rho_0(0.435275)$	$\rho=\rho_0$	2.0985569	8.883337134	105.744246 %
	0.0175	0.5	-1.2		$\rho=\rho_0(0.435275)$	$\rho=\rho_0$	1.96565377	8.425870472	107.039818 %
	0.07	0.5	-1.2		$\rho=\rho_0(0.435275)$	$\rho=\rho_0$	1.5669444	7.053470485	112.165436 %
	0	0	-3.5		$\rho=\rho_0(0.286974)$	$\rho=\rho_0$	9.76022621	13.14523571	16.0524324 %
	0.0175	0	-3.5		$\rho=\rho_0(0.286974)$	$\rho=\rho_0$	9.23347949	12.46829336	16.2039288 %
$0.5\le R\le 1$	0.07	0	-3.5		$\rho=\rho_0(0.286974)$	$\rho=\rho_0$	7.65323933	10.43746631	16.7817277 %
	0	0.25	-3		$\rho=\rho_0(0.343)$	$\rho=\rho_0$	7.72178021	12.19047619	25.6468535 %
	0.0175	0.25	-3		$\rho=\rho_0(0.343)$	$\rho=\rho_0$	7.31193343	11.56270125	25.7516053 %
	0.07	0.25	-3	$R_1=0.7$	$\rho=\rho_0(0.343)$	$\rho=\rho_0$	6.08239309	9.679376422	26.1497655 %
	0	0.5	-2.5		$\rho=\rho_0(0.409963)$	$\rho=\rho_0$	5.85442477	11.24856042	38.6137643 %
	0.0175	0.5	-2.5		$\rho=\rho_0(0.409963)$	$\rho=\rho_0$	5.54692806	10.66929147	38.6888423 %
	0.07	0.5	-2.5		$\rho=\rho_0(0.409963)$	$\rho=\rho_0$	4.62443793	8.93148461	38.9736135 %

Table 2. Angular speed required for initial yielding Ω_i^2 and fully plastic state Ω_f^2 of a rotating disc for *m*=0.00, 2.9 and $\Theta_1 = 0.00, 0.0175, 0.07$.

	Temperature Θ_1	Density variation	Yielding Occurs at	Rotating Disc having Variable density $\rho = \rho_0 (r/b)^{-m}$		Angular speed required for initial yielding	Angular speed required for fully plastic state	Percentage increase in Angular speed %
		m	R	$r=a$	$r=b$	Ω_i^2	Ω_f^2	$\left(\sqrt{\Omega_f^2/\Omega_i^2}-1\right)\times 100$
$0.5 \le R \le 1$	0 0.0175 0.07	0 0 0	$R_1 = 0.5$	$\rho=\rho_0$ $\rho=\rho_0$ $\rho=\rho_0$	$\rho=\rho_0$ $\rho=\rho_0$ $\rho=\rho_0$	2.42436611 4.11524145 3.39044371	4.8487322 4.569321 3.7310872	41.421356 % 5.37271587 % 4.90336764 %
	0 0.0175 0.07	2.9 2.9 2.9	$R_1 = 0.7$	$\rho=\rho_0(2.303151)$ $\rho=\rho_0(2.303151)$ $\rho=\rho_0(2.303151)$	$\rho=\rho_0$ $\rho=\rho_0$ $\rho=\rho_0$	1.05590319 0.99505611 0.81251484	2.986545235 3.024246132 3.137348825	68.17928 % 74.33508 % 96.50145 %

In Figs. 3(a)-3(c), curves have been drawn between stresses and radius ratio $R = r/b$ for fully plastic state at different values of m = 0, 2.9, 3.5. It is seen that from figs. 3(a) and 3(c), radial stresses is maximum at the internal surface whereas form fig. 3(b), the circumferential stresses is maximum at the outer surface of the rotating disc. With the introduction of thermal effect it decreases the value of radial and circumferential stresses at inner and outer surface for fully-plastic state.

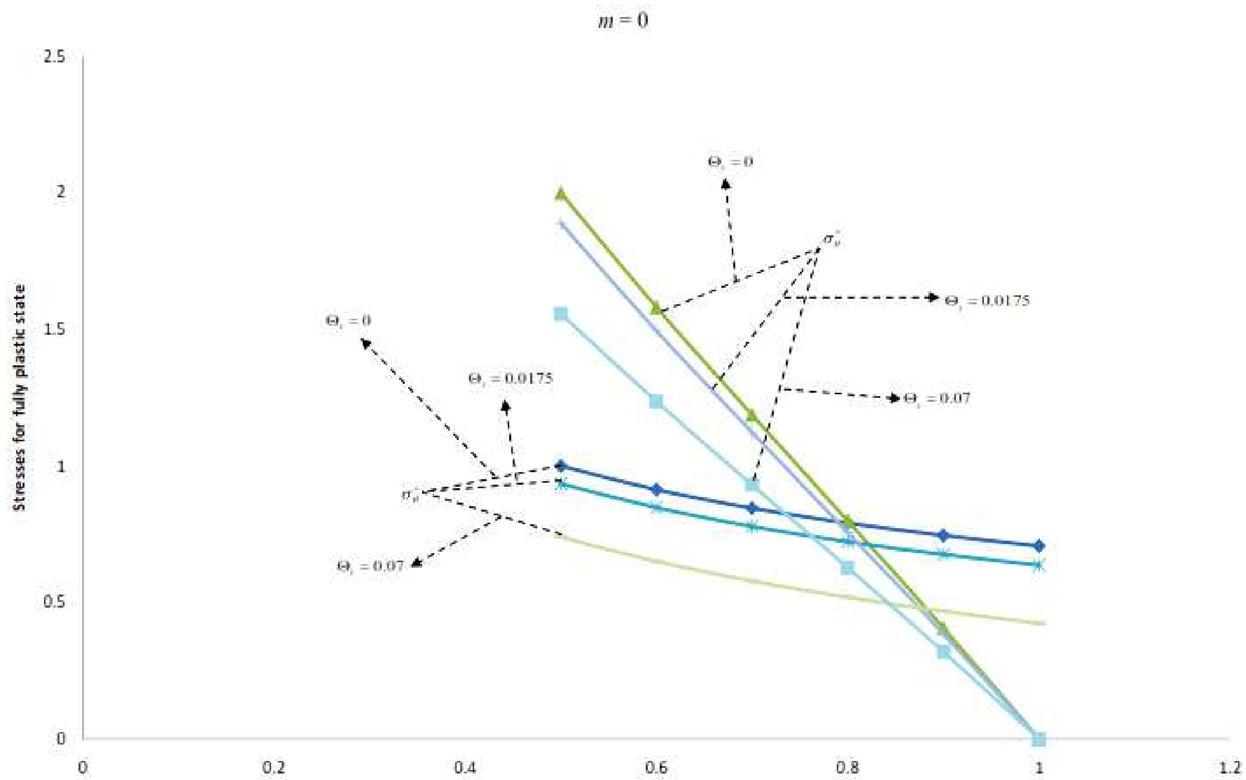

Figure 3(a). Stresses at fully-plastic state for different values of temperature and *m*=0.00 with respect to radii ratio R=r/b.

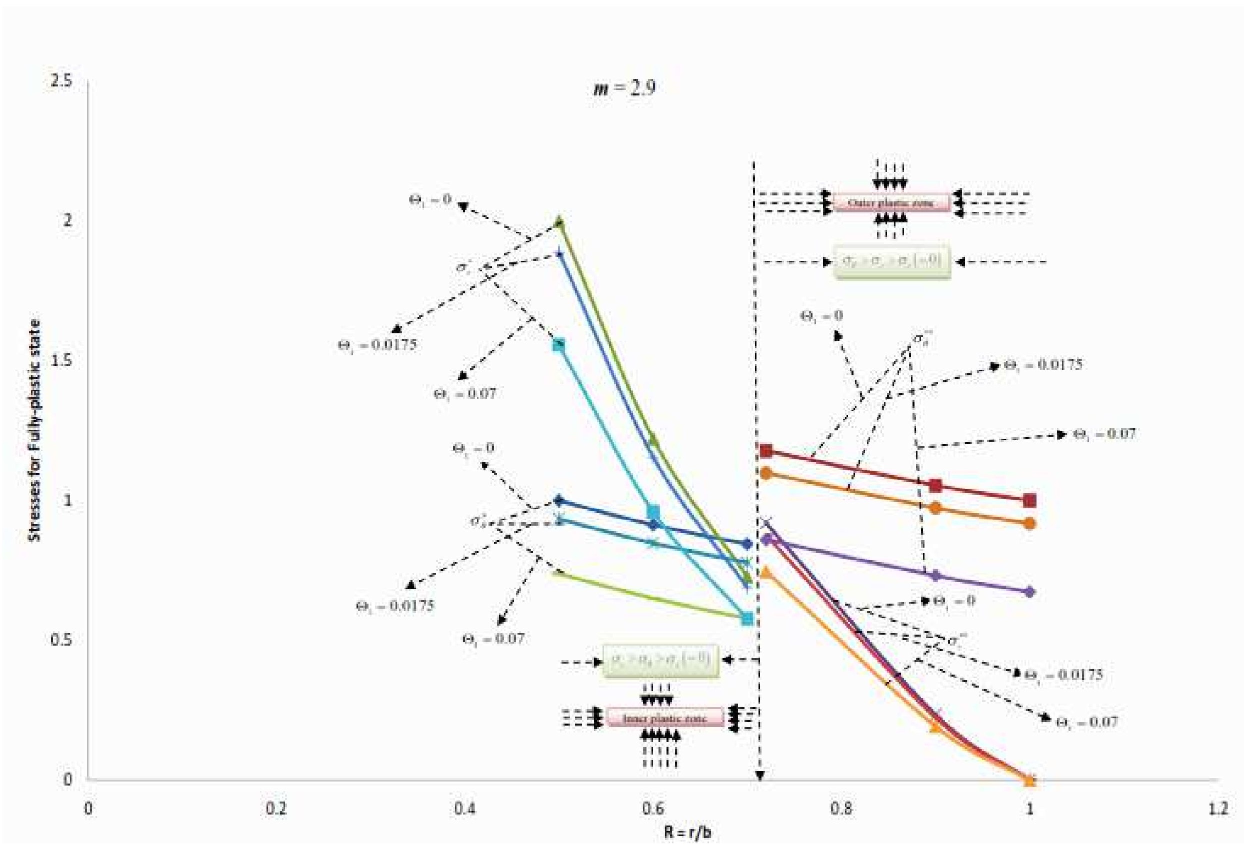

Figure 3(b). Stresses at fully-plastic state for different values of temperature and m=2.9 with respect to radii ratio R=r/b.

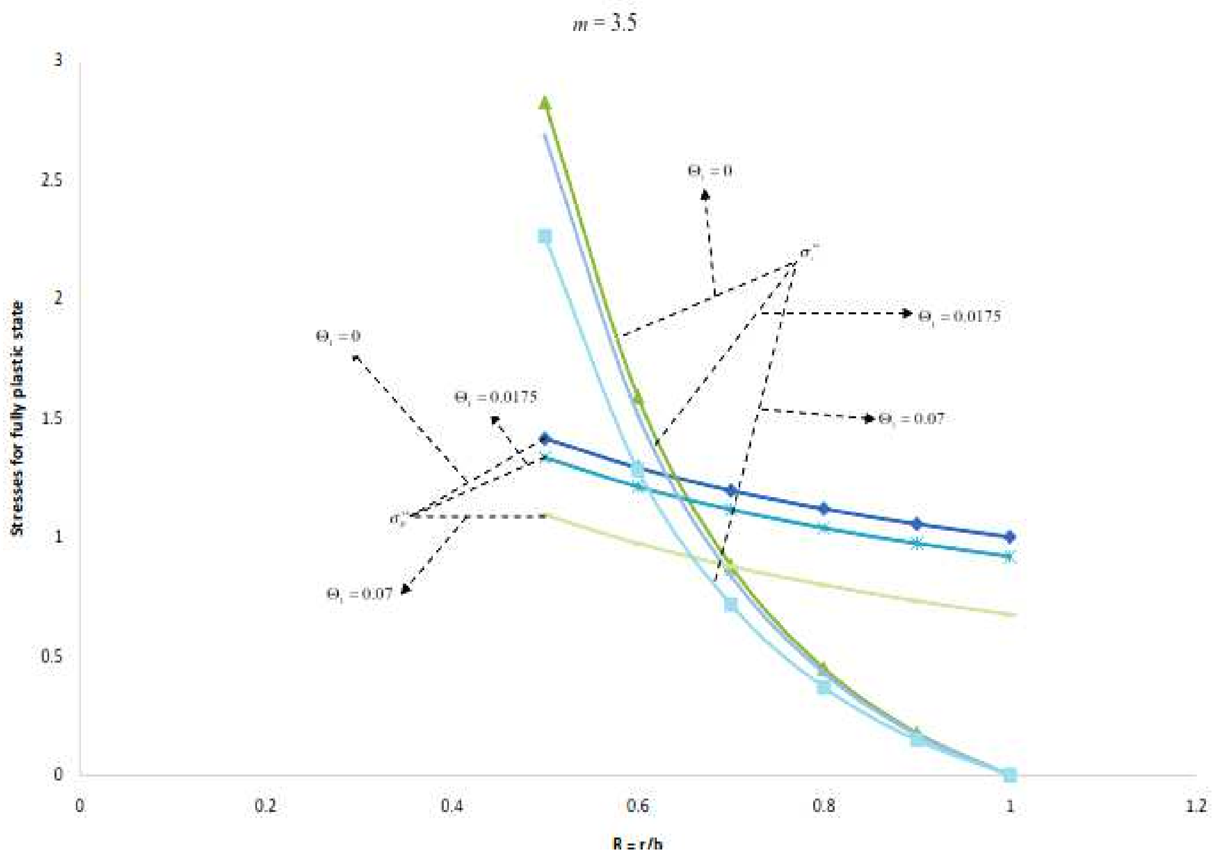

Figure 3(c). Stresses at fully-plastic state for different values of temperature and m=3.5 with respect to radii ratio R=r/b.

CONCLUSION

It has been seen that compressible material required higher percentage increased angular speed to become fully-plastic as compare to rotating disc made of incompressible material. Compressible material of rotating disc with shaft having variable density required higher percentage increased angular speed to become fully-plastic as compare to incompressible material. Circumferential stresses are maximal at the outer surface of the rotating disc. With effect of thermal load value of radial and circumferential stresses at inner and outer surface for fully-plastic state must be decrease.

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Nomenclature

a,b - Inner and outer radii of the disc [m],
ω - Angular velocity of rotation, [s^{-1}]
u,v,w - Displacement components, [m]
ρ - Density of material, [kgm^{-3}]
C - Compressibility, [-]
T_{ij}, e_{ij} -Stress [$kgm^{-1}s^{-2}$] and Strain rate tensor
Y - Yield stress, [$kgm^{-1}s^{-2}$]

Greek letters

$R = r / b; R_0 = a / b$ Radii ratio, [-]
σ_r - Radial stress component (T_{rr} / Y), [-]
σ_θ - Circumferential stress component ($T_{\theta\theta} / Y$), [-]
Θ - Temperature, [^0F]

A, B, d - Constants of integration, [-]

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