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STRESSES IN A THICK-WALLED CIRCULAR CYLINDER HAVING PRESSURE BY USING CONCEPT OF GENERALIZED STRAIN MEASURE

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ABSTRACT. Analysis of stress in a thick-walled circular cylinder subjected to uniform pressure is discussed by using Seth transition theory. It is shown that the circumferential stresses are maximal at the internal surface.

Key words: elastic-plastic, creep, transition, stresses, strain, pressure, cylinder.

INTRODUCTION

Problem of "Thick walled circular cylinders under internal pressure" has been discussed by many authors [3-5, 8-10] for isotropic plastic and creep theory. In their treatment the following assumptions were made:

(i) The incompressibility condition
(ii) Creep strain law
(iii) Yield condition.

Seth's transition theory [1] does not require any ad hoc assumptions like a yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deforming field and has been successfully applied to a large number of the problems in plasticity.

Seth has defined the generalized principal strain measure as:

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1 - 2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ii} = \frac{1}{n}\left[1 - \left(1 - 2\overset{A}{e}_{ii}\right)^{\frac{n}{2}}\right], \quad (i = 1, 2, 3) \tag{1}$$

where n is the measure and $\overset{A}{e}_{ii}$ is the principal finite strain components. In this paper we investigate the effect of stresses in a thick-walled circular cylinder having pressure by using concept of generalized strain measure by using Seth's transition theory.

Governing Equations

We consider a thick-walled circular cylinder of internal radius a and external radius b respectively subjected to internal pressure p . The displacement components in cylindrical polar co-ordinate are given by [2]:

$$u = r(1-\beta)\ ;\ v = 0\ ;\ w = dz \tag{2}$$

where β is function of $r = \sqrt{x^2 + y^2}$ only and d is a constant.
The finite strain components are given by Seth [1]:

$$\begin{aligned}
e_{rr} &= \frac{1}{2}\left[1-\left(r\beta'+\beta\right)^2\right]\\
e_{\theta\theta} &= \frac{1}{2}\left[1-\beta^2\right]\\
e_{ZZ} &= \frac{1}{2}\left[1-(1-d)^2\right]\\
\overset{A}{e}_{r\theta} &= \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0
\end{aligned} \tag{3}$$

where $\beta' = d\beta/dr$ and meaning of superscripts "A" is Almansi.

Substituting Eqs. (3) into Eq. (1), the generalized components of strain are:

$$\begin{aligned}
e_{rr} &= \frac{1}{n}\left[1-\left(r\beta'+\beta\right)^n\right]\\
e_{\theta\theta} &= \frac{1}{n}\left[1-\beta^n\right]\\
\mathrm{e}_{zz} &= \frac{1}{\mathrm{n}}\left[1-(1-\mathrm{d})^{\mathrm{n}}\right]\\
e_{r\theta} &= e_{\theta z} = e_{zr} = 0
\end{aligned} \tag{4}$$

The stress–strain relations for isotropic material are given by [19]:

$$T_{ij} = \lambda\,\delta_{ij}\,I_1 + 2\mu\, e_{ij} \quad , \quad (i, j = 1, 2, 3) \tag{5}$$

where T_{ij} and e_{ij} are stress and strain tensor respectively, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta.
Substituting the strain components from eq. (4) in eq. (5), the stresses are obtained as:

$$\begin{aligned}
T_{rr} &= \lambda I_1 + \frac{2\mu}{n}\left[1-(r\beta' + \beta)^n\right]\\
T_{\theta\theta} &= \lambda I_1 + \frac{2\mu}{n}\left[1-\beta^n\right]\\
T_{zz} &= \lambda I_1 + \frac{2\mu}{n}\left[1-(1-d)^n\right]\\
T_{r\theta} &= T_{\theta z} = T_{zr} = 0
\end{aligned} \tag{6}$$

where $I_1 = \frac{1}{n}\left[3-\left(r\beta'+\beta\right)^n-\beta^n-\left(1-d\right)^n\right]$.

The equations of equilibrium are all satisfied except:

$$\frac{d\,T_{rr}}{dr}+\frac{T_{rr}-T_{\theta\theta}}{r}=0\,. \tag{7}$$

Using Eqs. (6) in eq. (7), we get a non-linear differential equation in β as:

$$nP(P+1)^{n-1}\beta\frac{dP}{d\beta}+nP(P+1)^n+(1-c)nP-\left[1-(P+1)^n\right]c=0 \tag{8}$$

where $c=2\mu/\lambda+2\mu$ and $r\beta'=\beta P$

Eq. (8) shows that the transition points of β are $P=0,\, P\to-1,\pm\infty.\; P\to 0,$ which does not give anything important.

We shall now show that the transition state at $P\to-1$ through different transition function leads to plastic or creep states.

The boundary conditions are:

$$\begin{aligned} T_{rr} &= \text{-}p \quad \text{at } r=a \\ T_{rr} &= 0 \quad\ \ \text{at } r=b \end{aligned} \tag{9}$$

where p is pressure applied internal surface.

Transition through the Principal Stresses

For finding the plastic stresses, the transition function is taken through the principal stress (SETH [1, 2], GUPTA [8-10], THAKUR [12-15]) at the transition point $P\to\pm\infty$. The transition function R is defined as:

$$R=\left(2-c\right)+\left(1-c\right)\left[1-\left(1-d\right)^n\right]-\frac{nc}{2\mu}T_{rr}\,. \tag{10}$$

Taking the logarithmic differentiation of Eq. (10) with respect to β ,we get

$$\frac{d}{d\beta}(\log R)=\frac{n}{\beta}\frac{[(\beta+1)^n+(P+1)^{n-1}\beta\frac{dP}{d\beta}]+n(1-c)}{(1-c)+(P+1)^n}\,. \tag{11}$$

Substituting the value of $dP/d\beta$ from the Eq. (8) in Eq. (11) and taking the asymptotic value $P\to-1$, we get after integration

$$R=Ar^{c/(1-c)} \tag{12}$$

where A is a constant of integration can be determined by boundary condition.

From Eqs. (10) and (12) we have

$$T_{rr}=\frac{\left(2-c\right)Y}{\left(3-2c\right)c}\left[(3-2c)-(1-c)(1-d)^n-Ar^{\frac{c}{1-c}}\right] \tag{13}$$

where $Y=\frac{2\mu(3-2c)}{n(2-c)}$ is yield stress in tension is given by [1].

By substituting the boundary conditions (9) in Eq. (13), we get

$$A=\frac{1}{b^{c/(1-c)}}\left[(3-2c)-(1-c)(1-d)^n\right]$$

and

$$p=\frac{(2-c)Y}{(3-2c)c}\left[(3-2c)-(1-c)(1-d)^n\right]\left[(a/b)^{c/1-c}-1\right] \tag{14}$$

where A is constant of integration and can be determined by boundary condition and p is the pressure at which the plasticity sets in cylinder. Substituting the value of A in Eq. (13), we get

$$T_{rr}=\frac{(2-c)}{c(3-2c)}Y\left[\{(3-2c)-(1-c)(1-d)^n\}\left\{1-(r/b)^{c/(1-c)}\right\}\right] \tag{15}$$

By substituting Eq. (15) in Eq. (7), we get

$$T_{\theta\theta}=\frac{(2-c)}{(3-2c)c}Y\left[(3-2c)-(1-c)(1-d)^n\right]\ [1-\frac{1}{1-c}(r/b)^{\frac{c}{1-c}}\}] \tag{16}$$

From Eqs. (15) and (16) we get

$$T_{rr}-T_{\theta\theta}=\frac{(2-c)}{(3-2c)(1-c)}Y\left[(3-2c)-(1-c)(1-d)^n\right](r/b)^{\frac{c}{1-c}} \tag{17}$$

whereas Eq. (6) gives:

$$T_{zz}=\frac{1-c}{(2-c)}\left[T_{rr}+T_{\theta\theta}\right]+2Ye_{zz} \tag{18}$$

The resultant force transmitted by the wall in axial direction is equal $\Pi a^2 p$, that is

$$2\Pi\int_a^b rT_{zz}dr=\Pi a^2 p \tag{19}$$

Combining Eqs. (18) and (19),

$$e_{zz}=\frac{pa^2c}{2Y(2-c)(b^2-a^2)} \tag{20}$$

and by substituting Eq. (20) into Eq. (18),

$$T_{zz} = \frac{1-c}{(2-c)}\left[T_{rr} + T_{\theta\theta}\right] + \frac{pa^2c}{(2-c)(b^2-a^2)} \quad . \tag{21}$$

For fully plastic state, letting $c \to 0$, we have from Eq. (19):

$$e_{zz} = 0 \tag{22}$$

which implies that the cylinder does not change its length when it deforms permanently. Further it shows that the strain vanishes in axial direction, a condition which is generally assumed in the classical theory.

We introduce the following non-dimensional components as:

$$R = r/b, R_0 = a/b, \sigma_r = T_{rr}/Y, \sigma_\theta = T_{\theta\theta}/Y, \sigma_z = T_{zz}/Y, \ p_0 = p/Y$$

Elastic-plastic transitional stresses and pressure from Eqs. (15), (16), (21) and (14) in non-dimensional form become:

$$\sigma_r = \frac{(2-c)}{c(3-2c)}\left[\{(3-2c)-(1-c)(1-d)^n\}\left\{1-R^{c/(1-c)}\right\}\right] \tag{23}$$

$$\sigma_\theta = \frac{(2-c)}{(3-2c)c}\left[(3-2c)-(1-c)(1-d)^n\right]\ [1-\frac{1}{1-c}R^{\frac{c}{1-c}}\}] \tag{24}$$

$$\sigma_z = \frac{1-c}{(2-c)}\left[\sigma_r + \sigma_\theta\right] + \frac{p_0R_0^2c}{(2-c)(1-R_0^2)} \tag{25}$$

and

$$p_0 = \frac{(2-c)}{(3-2c)c}\left[(3-2c)-(1-c)(1-d)^n\right]\left[R_0^{\frac{c}{(1-c)}} - 1\right] \tag{26}$$

Using Eqs. (4) and (22), the stresses (23)-(25) and pressure (26) for fully plastic state become

$$\sigma_r = \log(1/R) \tag{27}$$

$$\sigma_\theta = \left[\log(1/R) - 1\right] \tag{28}$$

$$\sigma_z = \frac{1}{2}\left[2\log(1/R) - 1\right] \tag{29}$$

$$\sigma_{rr} - \sigma_{\theta\theta} = \frac{4}{3} \tag{30}$$

and $$p_0 = -\log\left(\frac{1}{R_0}\right) \tag{31}$$

From Eq. (30), we note that Tresca's yield condition comes out from the analysis itself. Expressions (29) for fully plastic state are the same as given by NADAI [16] and HILL [17] by assuming $e_{zz} = 0$ and Tresca's yield condition. MACGREHGER *et al.* [18] also arrived at this special solution for finite deformation.

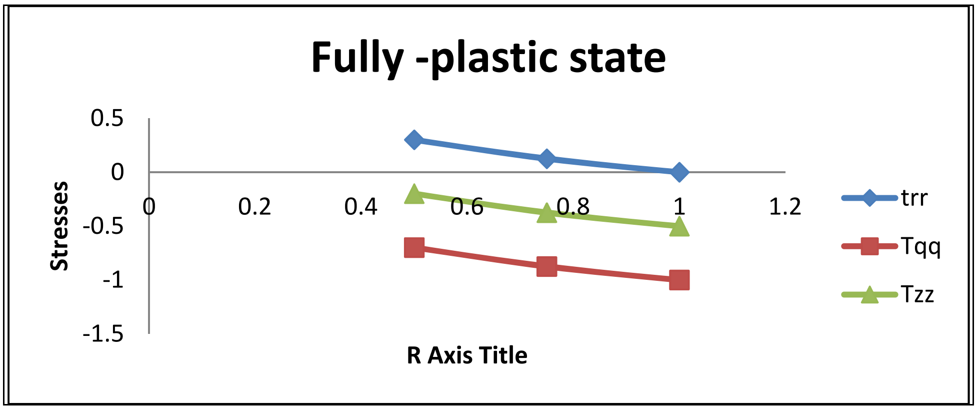

Figure 1. Stresses for fully-plastic state.

Solution through the Principal Stress difference

It has been shown that the asymptotic solution through the principal stress difference [8-15] at the transition point $P \to -1$, gives the creep stresses. We define the transition function R_1 as:

$$R_1 = T_{rr} - T_{\theta\theta} = \left(\frac{2\mu}{n}\right)\left[\beta^n - \left(r\beta' + \beta\right)^n\right] . \tag{32}$$

Taking the logarithmic differentiating of eq. (32) with respect to β and using eq. (8), we get:

$$\frac{d}{d\beta}(\log R_1) = \frac{n}{\beta}\left[\frac{nP(2-c) - c\{1-(P+1)^n\}}{nP(1-(P+1)^n)}\right] \tag{33}$$

Taking asymptotic value of eq. (32) at $P \to$-1, we get after integration:

$$R_1 = A_1 \beta^{2n-c(n-1)} \tag{34}$$

where A_1 is a constant of integration can be determined by boundary condition. The asymptotic value of β as is $P \to$-1 is D/r, D being a constant.
Eqs. (32) and (34) yield:

$$T_{rr} - T_{\theta\theta} = R_1 = A_1 r^{-2n+c(n-1)} \tag{35}$$

which substituted into Eq. (7) and by integrating yields

$$T_{rr} = A_1 \frac{r^{-2n+c(n-1)}}{2n-c(n-1)} + B_1 \tag{36}$$

where B_1 is a constant of integration can be determined by boundary condition.
Using the boundary condition (9) in Eq. (36), we get the transitional stresses for state of creep as:

$$T_{rr} = -p\left[\frac{(b/r)^{2n-c(n-1)} - 1}{(b/a)^{2n-c(n-1)} - 1}\right] \tag{37}$$

$$T_{\theta\theta} = -p \ \frac{\left[\left(1-2n+c(n-1)\right)\left(b/r\right)^{2n-c(n-1)} - 1\right]}{\left(b/a\right)^{2n-c(n-1)} - 1} \tag{38}$$

$$T_{zz} = \frac{1-c}{2-c}\left(T_{rr} + T_{\theta\theta}\right) + \frac{2\mu(3-2c)}{2-c} e_{zz} \tag{39}$$

Using Eqs. (19) and (37)-(38), we get:

$$\frac{2\mu(3-2c)}{\left(2-c\right)} e_{zz} = \frac{npa^2c}{2(2-c)} \frac{1}{(b^2-a^2)} \ . \tag{40}$$

When the material is incompressible i.e., $c \rightarrow 0$, Eq. (40) becomes:

$$e_{zz} = 0 \tag{41}$$

The steady state creep stresses (37)-(39) for incompressible material are given by:

$$T_{rr} = q\left(r^{-2n} - b^{-2n}\right) \tag{42}$$

$$T_{\theta\theta} = q \ \left[r^{-2n}\left(1-2n\right) - b^{-2n}\right] \tag{43}$$

$$T_{zz} = q \ \left[(r^{-2n} - b^{-2n}) - nr^{-2n}\right] \tag{44}$$

where $q = \dfrac{p}{(a^{-2n} - b^{-2n})}$

These expressions for steady state creep are the same as given by BAILEY [3] and ODQUIST [4]. They have derived these equation by assuming Norton's law, Von-Mises yield condition, and incompressibility of the material provided as put n =1/N

For $c \rightarrow 0$ and $n \rightarrow 0$ (*i.e.* for Hencky strain measure) Eq. (35) gives the Tresce condition,

$$T_{rr} - T_{\theta\theta} = A_1 \tag{45}$$

Numerical Results and Discussion:

When a material passes from elastic state to the plastic state or to the creep state, transition takes place . Since this transition is non linear and therefore difficult to investigate, workers have assumed certain ad hoc assumption like a strain law, incompressibility of the material and an yield condition which may or may not be relevant. Further they have also assumed different constitutive equations for the above mentioned three states. It has been shown in this paper that by using Seth's transition theory, there is no need to assume the above condition. On the other hand, these condition follow from the analysis itself, which can be seen from Eqs. (22), (30), (42), (43), (44) and (45). Further do not use different constitutive equations for each state. The solution thus obtained not only give the creep stresses for compressible material but also for incompressible material as a particular case. For elastic-plastic state, the transitional stresses are obtained and it has been shown from these stresses that as soon as the material become plastic, the stresses obtained are the same as in the classical theory which are based on certain ad hoc assumptions.

Curve have drawn from Fig. 2 between stresses and radius for fully plastic state, it has been seen that the radial stresses is maximum at the international surface.

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