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RELATIONS BETWEEN THE POLYNOMIALS OF A GRAPH AND WIENER INDEX

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ABSTRACT. Two new degree–based graph polynomials are introduced, and their relations to the Wiener index and degree distance (Schultz index) established. Expressions are obtained, enabling the calculation of these polynomials for graph products.

1 Introduction

In contemporary chemical graph theory, a large number of molecular structure descriptors are being considered, that depend on distances and vertex degrees of molecular graphs. Details on distance–based structure descriptors and their applications can be found in the books [10,11] and the references cited therein. Details of degree–based structure descriptors can be found in the recent papers [6,22] and the references cited therein.

The graphs considered in this paper (and in chemical graph theory in general [12]) are assumed to be connected and simple. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u and v of G is denoted by $d(u,v)$ and is defined as the number of edges in a shortest path connecting u and v [1]. The oldest and most studied distance–based structure descriptor is the Wiener index [3, 4, 14, 19], introduced as early as in 1947 [23] and defined as the sum of distances between all pairs of vertices of the underlying graph:

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u,v) \ .$$

In the 1980s Hosoya [17] came to the ingenious idea to use a polynomial to generate distance distributions for graphs. It was later recognized that equivalent results were much earlier put forward by Altenburg [15], but these had little impact on the development of the concept.

Hosoya defined the polynomial

$$Hos(G,x) = \sum_{k \geq 0} d(G,k)\, x^k$$

where $d(G,k)$ is the number of vertex pairs in the graph G whose distance is k. An alternative (but equivalent) way of writing this polynomial is

$$Hos(G,x) = \sum_{\{u,v\} \subset V(G)} x^{d(u,v)} \ . \tag{1.1}$$

The first derivative of $Hos(G,x)$ at $x = 1$ is equal to the Wiener index of G. For this reason Hosoya named this polynomial *Wiener polynomial*. Late the name *Hosoya polynomial* was proposed and this name prevails in the contemporary literature. Details on the theory of the Hosoya polynomial as well as an exhaustive bibliography can be found in the review [15].

The degree d_u of a vertex u is the number of its first neighbors. The oldest and most studied degree–based structure descriptors are the first and second Zagreb indices, defined as

$$M_1(G) = \sum_{v \in V(G)} d_v^2 \qquad \text{and} \qquad M_2(G) = \sum_{uv \in E(G)} d_u \, d_v \ .$$

For details see [2, 9, 18, 22].

Motivated by formula (1.1), three of the present authors [21] conceived analogous degree–based polynomials, namely

$$S_G(x) = \sum_{u \in V(G)} x^{d_u} \qquad \text{and} \qquad H_G(x) = \sum_{uv \in E(G)} x^{d_u + d_v}$$

and proved the following results:

Theorem 1.1. [21] Let G be a graph, not necessarily connected. Then,

$$S_G(1) = |V(G)| \quad ; \quad S'_G(1) = 2|E(G)| \quad ; \quad H_G(1) = |E(G)| \ .$$

We can now add to this:

Theorem 1.2. *Let G be a graph, not necessarily connected. Then,*

$$H'_G(1) = M_1(G)$$

with $M_1(G)$ denoting the first Zagreb index.

Proof. The first derivative of the polynomial $H_G(x)$ is equal to $\sum_{uv} (d_u + d_v) \, x^{d_u + d_v - 1}$ which for $x = 1$ yields $\sum_{uv} (d_u + d_v)$. It has been shown [6] that the first Zagreb index obeys the identity

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) \ .$$

□

Let G_1 and G_2 be two graphs with disjoint vertex sets. Then the graph products $G_1 + G_2$, $G_1 \times G_2$, and $G_1[G_2]$ are defined as follows [16]:

1. $G_1 + G_2$ is the graph for which $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$.

2. $G_1 \times G_2$ is the graph for which $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u, v)(u', v') \in E(G_1 \times G_2)$ if $u = u'$ and $vv' \in E(G_2)$ or $v = v'$ and $uu' \in E(G_1)$.

3. $G_1[G_2]$ is the graph for which $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and $(u,v)(u',v') \in E(G_1[G_2])$ if $u = u'$ and $vv' \in E(G_2)$ or $uu' \in E(G_1)$.

In [21] also the following result was proven:

Theorem 1.3. [21] Let for $i = 1, 2$, the number of vertices of the graph G_i be denoted by p_i . Then

$$H_{G_1+G_2}(x) = x^{2p_2}\, H_{G_1}(x) + x^{2p_1}\, H_{G_2}(x) + x^{p_1+p_2}\, S_{G_1}(x) \cdot S_{G_2}(x) \ .$$

In what follows we shall be concerned with a further structure descriptor, defined as

$$S(G) = \sum_{\{u,v\} \subset V(G)} (d_u + d_v)\, d(u,v) \ .$$

This degree–weighted version of the Wiener index was first time introduced by Dobrynin and Kochetova [5] and called *degree distance*.

The same quantity was examined in the paper [8] under the name *Schultz index*. Namely, somewhat earlier H. P. Schultz [20] proposed a structure descriptor named *molecular topological index*, defined as

$$MTI(G) = \sum_{i=1}^{n} \mathbf{d}(\mathbf{A} + \mathbf{D})_i$$

where $\mathbf{A}$ and $\mathbf{D}$ are the adjacency and distance matrices of the underlying molecular graph G, and $\mathbf{d}$ is the vector of vertex degrees. It can be easily shown that

$$MTI(G) = \sum_{u \in V(G)} d_u^2 + \sum_{\{u,v\} \subset V(G)} (d_u + d_v)\, d(u,v)$$

which means that

$$MTI(G) = M_1(G) + S(G) \ .$$

More details on the properties of degree distance can be found in a recent paper [7].

2 Main Results

Definition 2.1. *Let G be a graph, not necessarily connected. We define the polynomial $\mathcal{K}(G,x)$ as:*

$$\mathcal{K}(G,x) = \sum_{\substack{\{u,v\} \subset V(G) \\ u \neq v}} x^{d_u + d_v} \ .$$

Example 2.2. *Let R_k be a k-regular graph on p vertices. Let K_n , C_n , W_n , $K_{m,n}$, and P_n be the complete graph, cycle, wheel, complete bipartite graph, and path, respectively. Then*

$$
\begin{aligned}
\mathcal{K}(R_k, x) &= \binom{p}{2} x^{2k} \\
\mathcal{K}(K_{m,n}, x) &= \binom{m}{2} x^{2n} + \binom{n}{2} x^{2m} + mn\, x^{m+n} \\
\mathcal{K}(K_n, x) &= \binom{n}{2} x^{2n-2} \\
\mathcal{K}(W_n, x) &= \binom{n-1}{2} x^6 + (n-1)x^{n+2} \\
\mathcal{K}(C_n, x) &= \binom{n}{2} x^4 \\
\mathcal{K}(P_n, x) &= x^2 + 2(n-2)x^3 + \binom{n-2}{2} x^4 \ .
\end{aligned}
$$

Corollary 2.3. *Let G be a graph with p vertices and q edges. Then*

$$
\begin{aligned}
\mathcal{K}(G, 1) &= H(G, 1) = \binom{p}{2} \\
\mathcal{K}'(G, 1) &= 2q\,(p-1) \ .
\end{aligned}
$$

Theorem 2.4. *Let for $i = 1, 2$, the number of vertices and edges of the graph G_i be, respectively, p_i and q_i . Then*

$$
\begin{aligned}
\mathcal{K}(G_1 + G_2, x) &= x^{2p_2}\mathcal{K}(G_1, x) + x^{2p_1}\mathcal{K}(G_2, x) + x^{p_1+p_2} S_{G_1}(x) S_{G_2}(x) \quad (2.1) \\
\mathcal{K}(G_1 \times G_2, x) &= S_{G_2}(x^2)\,\mathcal{K}(G_1, x) + S_{G_1}(x^2)\,\mathcal{K}(G_2, x) \\
&+ 2\mathcal{K}(G_1, x)\,\mathcal{K}(G_2, x) \quad (2.2) \\
\mathcal{K}(G_1[G_2], x) &= S_{G_2}(x^2)\,\mathcal{K}(G_1, x^{p_2}) + S_{G_1}(x^{2p_2})\,\mathcal{K}(G_2, x) \\
&+ 2\mathcal{K}(G_1, x^{p_2})\,\mathcal{K}(G_2, x) \ . \quad (2.3)
\end{aligned}
$$

Proof. In what follows, we always assume that $u \neq v$.

Identity (2.1):

$$\mathcal{K}(G_1 + G_2, x) = \sum_{\{u,v\} \subset V(G_1+G_2)} x^{d_u+d_v} = \sum_{\{u,v\} \subset V(G_1)} x^{d_u+d_v+2p_2}$$

$$+ \sum_{\{u,v\}\subset V(G_2)} x^{d_u+d_v+2p_1} + \sum_{u\in V(G_1),\ v\in V(G_2)} x^{d_u+d_v+p_1+p_2}$$

$$= x^{2p_2} \sum_{\{u,v\}\subset V(G_1)} x^{d_u+d_v} + x^{2p_1} \sum_{\{u,v\}\subset V(G_2)} x^{d_u+d_v} + x^{p_1+p_2} \sum_{u\in V(G_1)} x^{d_u} \sum_{v\in V(G_2)} x^{d_v}$$

$$= x^{2p_2}\mathcal{K}(G_1,x) + x^{2p_1}\mathcal{K}(G_2,x) + x^{p_1+p_2} S_{G_1}(x) S_{G_2}(x)\ .$$

Identity (2.2):

$$\mathcal{K}(G_1 \times G_2, x) = \sum_{\{u,v\}\subset V(G_1\times G_2)} x^{d_u+d_v} = \sum_{\substack{\{u_1,v_1\}\subset V(G_1)\\ u_2=v_2\in V(G_2)}} x^{d_{u_1}+d_{v_1}+2d_{u_2}}$$

$$+ \sum_{\substack{\{u_2,v_2\}\subset V(G_2)\\ u_1=v_1\in V(G_1)}} x^{d_{u_2}+d_{v_2}+2d_{u_1}} + \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1, u2\neq v2}} x^{d_{u_1}+d_{v_1}+d_{u_2}+d_{v_2}}$$

$$= \sum_{u_2\in V(G_2)} (x^2)^{d_{u_2}} \sum_{\{u_1,v_1\}\subset V(G_1)} x^{d_{u_1}+d_{v_1}} + \sum_{u_1\in V(G_1)} (x^2)^{d_{u_1}} \sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}}$$

$$+ \ 2 \sum_{\{u_1,v_1\}\subset V(G_1)} x^{d_{u_1}+d_{v_1}} \sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}}$$

$$= S_{G_2}(x^2)\,\mathcal{K}(G_1,x) + S_{G_1}(x^2)\,\mathcal{K}(G_2,x) + 2\mathcal{K}(G_1,x)\,\mathcal{K}(G_2,x)\ .$$

Identity (2.3):

$$\mathcal{K}(G_1[G_2], x) = \sum_{\{u,v\}\subset V(G_1[G_2])} x^{d_u+d_v} = \sum_{\substack{\{u_1,v_1\}\subset V(G_1)\\ u_2=v_2\in V(G_2)}} x^{p_2(d_{u_1}+d_{v_1})+2d_{u_2}}$$

$$+ \sum_{\substack{\{u_2,v_2\}\subset V(G_2)\\ u_1=v_1\in V(G_1)}} x^{d_{u_2}+d_{v_2}+2p_2d_{u_1}} + \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1[G_2])\\ u_1\neq v_1, u2\neq v2}} x^{p_2(d_{u_1}+d_{v_1})+d_{u_2}+d_{v_2}}$$

$$= \sum_{u_2\in V(G_2)} (x^2)^{d_{u_2}} \sum_{\{u_1,v_1\}\subset V(G_1)} (x^{p_2})^{d_{u_1}+d_{v_1}} + \sum_{u_1\in V(G_1)} (x^{2p_2})^{d_{u_1}} \sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}}$$

$$+ \ 2 \sum_{\{u_1,v_1\}\subset V(G_1)} x^{p_2(d_{u_1}+d_{v_1})} \sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}}$$

$$= S_{G_2}(x^2)\,\mathcal{K}(G_1,x^{p_2}) + S_{G_1}(x^{2p_2})\,\mathcal{K}(G_2,x) + 2\mathcal{K}(G_1,x^{p_2})\,\mathcal{K}(G_2,x)\ .$$

□

Corollary 2.5. *Let G_1 and G_2 be same as in Theorem 2.4 Then by Theorem (2.4) and corollary (2.3) for the graphs $G_1 + G_2$ and $G_1 \times G_2$ we have:*

$$\begin{aligned} \mathcal{K}(G_1 + G_2, 1) &= \mathcal{K}(G_1, 1) + \mathcal{K}(G_2, 1) + p_1\, p_2 \\ \mathcal{K}(G_1 \times G_2, 1) &= \mathcal{K}(G_1[G_2], 1) = p_2\, \mathcal{K}(G_1, 1) + p_1\, \mathcal{K}(G_2, 1) + 2\mathcal{K}(G_1, 1)\, \mathcal{K}(G_2, 1)\ . \end{aligned}$$

Definition 2.6. *Let G be a graph. We define the polynomial $\mathcal{F}(G, x)$ as:*

$$\mathcal{F}(G, x) = \sum_{\{u,v\} \subset V(G)} d(u, v)\ x^{d_u + d_v}$$

Example 2.7. *Using the same notation as in Example 2.2, we have:*

$$\begin{aligned} \mathcal{F}(C_{2n}, x) &= n^3\, x^4 \\ \mathcal{F}(C_{2n+1}, x) &= \frac{n(n+1)(2n+1)}{2}\, x^4 \\ \mathcal{F}(K_{m,n}, x) &= 2\binom{m}{2} x^{2n} + 2\binom{n}{2} x^{2m} + mn\, x^{m+n} \\ \mathcal{F}(W_n, x) &= (n-3)(n-1)\, x^6 + (n-1)x^{n+2} \\ \mathcal{F}(P_n, x) &= (n-1)\, x^2 + 2\binom{n-1}{2} x^3 + \binom{n-1}{3} x^4\ . \end{aligned}$$

Corollary 2.8. *Let G be a connected graph. Then,*

$$\mathcal{F}(G, 1) = W(G) \qquad , \qquad \mathcal{F}'(G, 1) = S(G)\ .$$

By the above corollary, the following previously known formulas are obtained:

$$\begin{aligned} &W(K_n) = \binom{n}{2} && S(K_n) = \binom{n}{2}(2n-2) \\ &W(C_{2n}) = n^3 && S(C_{2n}) = 4n^3 \\ &W(C_{2n+1}) = n(n+1)(2n+1)/2 && S(C_{2n+1}) = 2n(n+1)(2n+1) \\ &W(K_{m,n}) = 2\binom{m}{2} + 2\binom{n}{2} + mn && S(K_{m,n}) = 4n\binom{m}{2} + 4m\binom{n}{2} + mn(m+n) \\ &W(W_n) = (n-1)(n-2) && S(W_n) = 2(n-1)(3n-8) \\ &W(P_n) = (n-1)^2 + \binom{n-1}{3} && S(P_n) = 3(n-1)^2 - (n-1) + 4\binom{n-1}{3} \end{aligned}$$

If $d(u,v) \leq 2$ holds for all $u,v \in V(G)$, then the graph G is said to be of diameter two.

Lemma 2.9. *Let G be a graph of diameter two. Then $\mathcal{F}(G,x) = 2\mathcal{K}(G,x) - H_G(x)$.*

Proof. Since G is of diameter two, $d(u,v) \leq 2$. It is clear that there are $|E(G)|$ vertex pairs such that $d(u,v) = 1$. Therefore,

$$\begin{aligned}
\mathcal{F}(G,x) &= \sum_{\{u,v\}\subset V(G)} d(u,v)\, x^{d_u+d_v} = \sum_{\substack{\{u,v\}\subset V(G)\\ d(u,v)=2}} 2\, x^{d_u+d_v} + \sum_{\substack{\{u,v\}\subset V(G)\\ d(u,v)=1}} x^{d_u+d_v} \\
&= \sum_{\{u,v\}\subset V(G)} 2\, x^{d_u+d_v} - \sum_{uv\in E(G)} x^{d_u+d_v} = 2\mathcal{K}(G,x) - H_G(x)\ .
\end{aligned}$$

□

Corollary 2.10. [13] *Let G be of diameter two. The Wiener index of G is given by*

$$W(G) = 2\binom{|G|}{2} - |E(G)|\ .$$

Theorem 2.11. *Let G_1 and G_2 be two graphs. Then, using the same notation as in Theorem 2.4,*

$$\begin{aligned}
\mathcal{F}(G_1+G_2,x) &= x^{2p_2}\Big(2\mathcal{K}(G_1,x) - H_{G_1}(x)\Big) + x^{2p_1}\Big(2\mathcal{K}(G_2,x) - H_{G_2}(x)\Big) \\
&+ x^{p_1+p_2}\, S_{G_1}(x)\, S_{G_2}(x) \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(G_1[G_2],x) &= S_{G_2}(x^2)\, \mathcal{F}(G_1,x^{p_2}) + S_{G_1}(x^{2p_2})\Big[2\mathcal{K}(G_2,x) - H_{G_2}(x)\Big] \\
&+ 2\mathcal{F}(G_1,x^{p_2})\, \mathcal{K}(G_2,x)\ . \qquad (2.5)
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(G_1\times G_2,x) &= S_{G_2}(x^2)\, \mathcal{F}(G_1,x) + S_{G_1}(x^2)\, \mathcal{F}(G_2,x) \\
&+ 2\Big[\mathcal{K}(G_2,x)\, \mathcal{F}(G_1,x) + \mathcal{K}(G_1,x)\, \mathcal{F}(G_2,x)\Big]\ . \qquad (2.6)
\end{aligned}$$

Proof. Identity (2.4): It is clear that the graph $G_1 + G_2$ always is of diameter two. Therefore from Lemma (2.9) and Eq. (2.1) it follows

$$\mathcal{F}(G_1+G_2,x) = 2\mathcal{K}(G_1+G_2,x) - H_{G_1+G_2}(x)$$

$$\begin{aligned}
&= 2\Big(x^{2p_2}\mathcal{K}(G_1,x) + x^{2p_1}\mathcal{K}(G_2,x) + x^{p_1+p_2}S_{G_1}(x)\,S_{G_2}(x)\Big)\\
&- \Big(x^{2p_2}H_{G_1}(x) + x^{2p_1}H_{G_2}(x) + x^{p_1+p_2}S_{G_1}(x)\,S_{G_2}(x)\Big)\\
&= x^{2p_2}\Big(2\mathcal{K}(G_1,x) - H_{G_1}(x)\Big)+x^{2p_1}\Big(2\mathcal{K}(G_2,x) - H_{G_2}(x)\Big)\\
&+ x^{p_1+p_2}S_{G_1}(x)\,S_{G_2}(x)\ .
\end{aligned}$$

Identity (2.5):

$$\begin{aligned}
\mathcal{F}(G_1[G_2],x) &= \sum_{\{u,v\}\subset V(G_1[G_2])} d(u,v)\,x^{d_u+d_v} = \sum_{\substack{\{u_1,v_1\}\subset V(G_1)\\ u_2=v_2\in V(G_2)}} d(u_1,v_1)\,x^{p_2(d_{u_1}+d_{v_1})+2d_{u_2}}\\
&+ \sum_{\substack{\{u_2,v_2\}\subset V(G_2)\\ u_1=v_1\in V(G_1)}} d(u_2,v_2)\,x^{d_{u_2}+d_{v_2}+2p_2d_{u_1}}\\
&+ \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1[G_2])\\ u_1\neq v_1,u2\neq v2)}} d(u,v)\,x^{p_2(d_{u_1}+d_{v_1})+d_{u_2}+d_{v_2}}\\
&= \sum_{u_2\in V(G_2)}(x^2)^{d_{u_2}}\sum_{\{u_1,v_1\}\subset V(G_1)} d(u_1,v_1)\,(x^{p_2})^{d_{u_1}+d_{v_1}}\\
&+ \sum_{u_1\in V(G_1)}(x^{2p_2})^{d_{u_1}}\Big[2\sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}} - \sum_{u_2v_2\in E(G_2)} x^{d_{u_2}+d_{v_2}}\Big]\\
&+ 2\sum_{\{u_1,v_1\}\subset V(G_1)} d(u_1,v_1)\,(x^{p_2})^{d_{u_1}+d_{v_1}}\sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}}\\
&= S_{G_2}(x^2)\mathcal{F}(G_1,x^{p_2}) + S_{G_1}(x^{2p_2})\Big[2\,\mathcal{K}(G_2,x) - H_{G_2}(x)\Big]+2\,\mathcal{F}(G_1,x^{p_2})\mathcal{K}(G_2,x)\ .
\end{aligned}$$

Identity (2.6):

$$\begin{aligned}
\mathcal{F}(G_1\times G_2,x) &= \sum_{\{u,v\}\subset V(G_1\times G_2)} d(u,v)\,x^{d_u+d_v} = \sum_{\substack{\{u_1,v_1\}\subset V(G_1)\\ u_2=v_2\in V(G_2)}} d(u_1,v_1)\,x^{d_{u_1}+d_{v_1}+2d_{u_2}}\\
&+ \sum_{\substack{\{u_2,v_2\}\subset V(G_2)\\ u_1=v_1\in V(G_1)}} d(u_2,v_2)\,x^{d_{u_2}+d_{v_2}+2d_{u_1}} + \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1,u2\neq v2)}} d(u,v)\,x^{d_u+d_v}\\
&= \sum_{u_2\in V(G_2)}(x^2)^{d_{u_2}}\sum_{\{u_1,v_1\}\subset V(G_1)} d(u_1,v_1)\,x^{d_{u_1}+d_{v_1}} + \sum_{u_1\in V(G_1)}(x^2)^{d_{u_1}}\sum_{\{u_2,v_2\}\subset V(G_2)} d(u_2,v_2)\,x^{d_{u_2}+d_{v_2}}\\
&+ \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1,u2\neq v2)}} d(u,v)\,x^{d_u+d_v}
\end{aligned}$$

$$= S_{G_2}(x^2)\mathcal{F}(G_1,x) + S_{G_1}(x^2)\mathcal{F}(G_2,x) + \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1, u2\neq v2)}} d(u,v)\, x^{d_u+d_v}\ .$$

Since the graph $G_1 \times G_2$ is connected, for each $u, v \in V(G_1 \times G_2)$ such that $u = (u_1, u_2), v = (v_1, v_2)$ and $u_1 \neq v_1, u_2 \neq v_2$, we consider the path $(u_1, u_2) \to (v_1, u_2) \to (v_1, v_2)$. Therefore,

$$\begin{aligned}
\sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1, u2\neq v2)}} d(u,v)\, x^{d_u+d_v} &= \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1, u2\neq v2)}} \Big[d(u_1,v_1) + d(u_2,v_2)\Big]\, x^{d_{u_1}+d_{v_1}+d_{u_2}+d_{v_2}} \\
&= \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1, u2\neq v2)}} d(u_1,v_1)\, x^{d_{u_1}+d_{v_1}+d_{u_2}+d_{v_2}} \\
&+ \sum_{\substack{\{(u_1,u_2),(v_1,v_2)\}\subset V(G_1\times G_2)\\ u_1\neq v_1, u2\neq v2)}} d(u_2,v_2)\, x^{d_{u_1}+d_{v_1}+d_{u_2}+d_{v_2}} \\
&= 2 \sum_{\{u_2,v_2\}\subset V(G_2)} x^{d_{u_2}+d_{v_2}} \sum_{\{u_1,v_1\}\subset V(G_1)} d(u_1,v_1)\, x^{d_{u_1}+d_{v_1}} \\
&+ 2 \sum_{\{u_1,v_1\}\subset V(G_1)} x^{d_{u_1}+d_{v_1}} \sum_{\{u_2,v_2\}\subset V(G_2)} d(u_2,v_2)\, x^{d_{u_2}+d_{v_2}} \\
&= 2\Big[\mathcal{K}(G_2,x)\mathcal{F}(G_1,x) + \mathcal{K}(G_1,x)\mathcal{F}(G_2,x)\Big]\ .
\end{aligned}$$

From the two above relations, Eq. (2.6) follows straightforwardly. □

Corollary 2.12. [24] *Using the same notation as in Theorem 2.4, the Wiener indices of $G_1 + G_2$, $G_1[G_2]$, and $G_1 \times G_2$ are given by:*

$$\begin{aligned}
W(G_1 + G_2) &= 2\binom{p_1}{2} + 2\binom{p_2}{2} + p_1 p_2 - (q_1 + q_2) \\
W(G_1[G_2]) &= p_1 \left[2\binom{p_2}{2} - q_2\right] + {p_2}^2 W(G_1) \\
W(G_1 \times G_2) &= {p_2}^2\, W(G_1) + {p_1}^2\, W(G_2)\ .
\end{aligned}$$

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