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THE MINIMUM COVERING ENERGY OF A GRAPH

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ABSTRACT. In this paper we introduce a new kind of graph energy, the minimum covering energy, $E_c(G)$. It depends both on the underlying graph G, and on its particular minimum cover C. Upper and lower bounds for $E_c(G)$ are established. The minimum covering energies of a number of well-known and much studied families of graphs are computed.

1 Introduction

The energy of a graph can be traced back to the 1930s, in which time the German scholar Erich Hückel put forward a method for finding approximate solutions of the Schrödinger equation of a class of organic molecules, the so-called unsaturated conjugated hydrocarbons [1, 2]. This approach is nowadays referred to as the Hückel molecular orbital (HMO) theory [3, 4].

Within HMO theory, the total energy of π-electrons is equal to the sum of the energies of all π-electrons in the considered molecule. It can be calculated from the eigenvalues of the underlying molecular graph [3, 5, 6]. Motivated by HMO total π-electron energy, one of the present authors [7] conceived the *energy of a graph*, defined as the sum of the absolute values of all graph eigenvalues. This definition is by no means restricted to molecular graphs, and enabled one to obtain a remarkable number and variety of novel mathematical results. For further information on the theory of graph energy refer to [8–11].

In connection with graph energy (that is defined in terms of the eigenvalues of the adjacency matrix), energy–like quantities were considered also for other matrices: Laplacian [12], distance [13], incidence [14], etc. [15–17]. Recall that a great variety of matrices has so far been associated with graphs [18]. In this paper we introduce a new matrix, called *minimum covering matrix* of a graph, and study its eigenvalues and energy.

All the graphs considered in this paper are finite, simple and undirected. In particular, these graphs do not possess loops. Let G be such a graph, of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. A subset C of V is called a *covering set* of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a *minimum covering set*. Let C be a minimum covering set of a graph G. The *minimum covering matrix* of G is the $n \times n$ matrix $A_c(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The characteristic polynomial of $A_c(G)$ is denoted by

$$f_n(G, \lambda) := \det(\lambda I - A_c(G)) \ .$$

The *minimum covering eigenvalues* of the graph G are the eigenvalues of $A_c(G)$. Since $A_c(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The *minimum covering energy* of G is then defined as

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{1.2}$$

In this paper, we are interested in studying mathematical aspects of the minimum covering energy of a graph. It is possible that the minimum covering energy that we are considering in this paper may have some applications in chemistry as well as in other areas.

The paper is organized as follows. In Section 2 we discuss some basic properties of minimum covering energy and derive an upper bound and a lower bound for $E_c(G)$. In Section 3 we compute the minimum covering energies of (i) star graphs, (ii) complete graphs, (iii) complete bipartite graphs, (iv) crown graphs, and (v) cocktail party graphs.

2 A chemical connection

The formulas (1.1) by which the minimum covering matrix is defined, can be viewed also as the definition of the ordinary adjacency matrix of a graph with loops. Indeed, $A_c(G)$ is the adjacency matrix of a graph, obtained from G by attaching a loop of weight $+1$ to each of its vertices belonging to the cover C.

Graphs with loops are the natural representations of heteroconjugated molecules, and have been much studied in chemical graph theory. In particular, rules for constructing their characteristic polynomials were elaborated in due detail [19–27]. Loops of weight $+1$ are just the graph representation of nitrogen atoms.

The HMO theory of graphs with loops (i. e., molecular graphs of heteroconjugated molecules) were also studied in detail, including the total π-electron energy [28–30].

All these results can be directly applied to the presently introduced minimal covering eigenvalues and minimal covering energy. For instance, based on a result from [30], the minimal covering energy, as defined by Eq. (1.2), can be represented by a Coulson–type integral formula:

$$E_c(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left[n - \frac{ix\, f_n'(G, ix)}{f_n(G, ix)} \right] dx$$

where $i = \sqrt{-1}$. This formula holds for any covering set C, that is, for any value of $|C|$.

3 Some basic properties of minimum covering energy

We first compute the minimum covering energy of two graphs, depicted in Fig. 1.

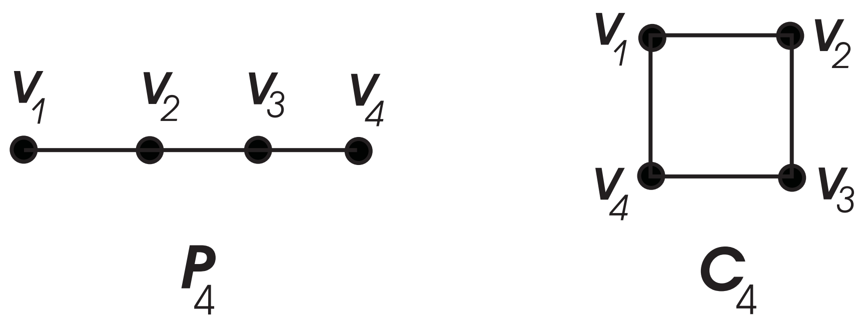

Fig. 1. Graphs considered in Examples 3.1 and 3.2.

Example 3.1. *Let G be the 4-vertex path P_4, with vertices v_1, v_2, v_3, v_4 (see Fig. 1), and let its minimum covering set be $C = \{v_1, v_3\}$. Then*

$$A_c(P_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $A_c(P_4)$ is $\lambda^4 - 2\lambda^3 - 2\lambda^2 + 3\lambda + 1$, the minimum covering eigenvalues are $(1-\sqrt{7+2\sqrt{5}})/2$, $(1+\sqrt{7+2\sqrt{5}})/2$, $(1-\sqrt{7-2\sqrt{5}})/2$, $(1+\sqrt{7-2\sqrt{5}})/2$, and therefore the minimum covering energy is

$$E_c(P_4) = \sqrt{7+2\sqrt{5}} + \sqrt{7-2\sqrt{5}}\,.$$

If we take another minimum covering set, namely $C^* = \{v_2, v_3\}$, then

$$A_{c^*}(P_4) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of $A_{c^*}(P_4)$ is $\lambda^4 - 2\lambda^3 - 2\lambda^2 + 2\lambda + 1$, the minimum covering eigenvalues are 1, -1, $1-\sqrt{2}$, $1+\sqrt{2}$, and this time the minimum covering energy is equal to $2+2\sqrt{2}$.

Example 3.1 illustrates the fact that the minimum covering energy of a graph G depends on the choice of the minimum covering set. i. e., that the minimum covering energy is not a graph invariant.

Example 3.2. *Let G be a cycle C_4 on 4 vertices v_1, v_2, v_3, v_4 (see Fig. 1), with minimum covering set $C = \{v_1, v_3\}$. Then*

$$A_c(C_4) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} .$$

The characteristic polynomial of $A_c(C_4)$ is $\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda$, the minimum covering eigenvalues are 0, 1, $(1+\sqrt{17})/2$, $(1-\sqrt{17})/2$, and thus the minimum covering energy is $E_c(C_4) = 1 + \sqrt{17}$.

Theorem 3.3. *Let G be a graph with vertex set V, edge set E, and a minimum covering set C. Let $f_n(G,\lambda) = c_0\,\lambda^n + c_1\,\lambda^{n-1} + c_2\,\lambda^{n-2} + \cdots + c_n$ be the characteristic polynomial of G. Then*

(i) $c_0 = 1$,

(ii) $c_1 = -|C|$,

(iii) $c_2 = \binom{|C|}{2} - |E|$, *and*

(iv) $c_3 = |C||E| - \sum_{v\in C} d(v) - \binom{|C|}{3} - 2\Delta$, *where Δ is the number of triangles in G.*

Proof. (i) Directly from the definition of $f_n(G,\lambda)$, it follows that $c_0 = 1$.

(ii) Since the sum of diagonal elements of $A_c(G)$ is equal to $|C|$, the sum of determinants of all 1×1 principal submatrices of $A_c(G)$ is the trace of $A_c(G)$, which evidently is equal to $|C|$. Thus, $(-1)^1\,c_1 = |C|$.

(iii) $(-1)^2\,c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_c(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1\le i<j\le n} (a_{ii}\,a_{jj} - a_{ij}\,a_{ji}) \\ &= \sum_{1\le i<j\le n} a_{ii}\,a_{jj} - \sum_{1\le i<j\le n} a_{ij}^2 = \binom{|C|}{2} - |E| . \end{aligned}$$

(iv) We have

$$\begin{aligned}
c_3 &= (-1)^3 \sum_{1\le i<j<k\le n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} \\
&= -\sum_{1\le i<j<k\le n} a_{ii}\left[(a_{jj}\,a_{kk} - a_{kj}\,a_{jk}) - a_{ij}\,(a_{ji}\,a_{kk} - a_{ki}\,a_{jk})\right. \\
&+ \left. a_{ik}\,(a_{ji}\,a_{kj} - a_{ki}\,a_{jj})\right] \\
&= -\sum_{1\le i<j<k\le n} a_{ii}\,a_{jj}\,a_{kk} + \sum_{1\le i<j<k\le n} [a_{ii}\,a_{jk} + a_{jj}\,a_{ik} + a_{kk}\,a_{ij}] \\
&- \sum_{1\le i<j<k\le n} a_{ij}\,a_{jk}a_{ki} - \sum_{1\le i<j<k\le n} a_{ik}\,a_{kj}\,a_{ji} \\
&- \binom{|C|}{3} + \sum_{1\le i<j<k\le n} [a_{ii}\,a_{jk} + a_{jj}\,a_{ik} + a_{kk}\,a_{ij}] - 2\Delta \\
&= -\binom{|C|}{3} + \left[\sum_{i=1}^{n} a_{ii}\right]\left[\sum_{1\le j<k\le n} a_{jk}\right] - \sum_{i=1}^{n} a_{ii} \sum_{\substack{k=1\\k\ne i}}^{n} a_{ik} - 2\Delta \ .
\end{aligned}$$

Thus

$$c_3 = |C||E| - \sum_{v\in C} d(v) - \binom{|C|}{3} - 2\Delta \ .$$

□

Theorem 3.4. *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of* $A_c(G)$ *, then*

$$\sum_{i=1}^{n} \lambda_i^2 = 2|E| + |C| \ .$$

Proof. The sum of squares of the eigenvalues of $A_c(G)$ is just the trace of $A_c(G)^2$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\,a_{ji} \\
&= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2 = 2|E| + |C| \ .
\end{aligned}$$

□

Bounds for $E_c(G)$, similar to McClelland's inequalities [31] for graph energy, are given in the following two theorems.

Theorem 3.5 (Upper bound)**.** *Let G be a graph with n vertices, m edges, and let C be a minimum covering set of G. Then*

$$E_c(G) \leq \sqrt{n(2m + |C|)}\ .$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A_c(G)$. Bearing in mind the Cauchy–Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

we choose $a_i = 1$ and $b_i = |\lambda_i|$, which by Theorem 3.4 implies

$$E_c(G)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \leq n\left(\sum_{i=1}^{n} |\lambda_i|^2\right) = n\sum_{i=1}^{n} \lambda_i^2 = n(2m + |C|)\ .$$

This completes the proof. □

Theorem 3.6 (Lower bound)**.** *Let G be a graph with n vertices and m edges, and let C be a minimum covering set of G. If $D = |detA_c(G)|$, then*

$$E_c(G) \geq \sqrt{2m + |C| + n(n-1)D^{2/n}}\ .$$

Proof.

$$[E_c(G)]^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)\left(\sum_{j=1}^{n} |\lambda_j|\right) = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i\neq j} |\lambda_i||\lambda_j|\ .$$

Employing the inequality between the arithmetic and geometric means, we obtain

$$\frac{1}{n(n-1)}\sum_{i\neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}\ .$$

Thus

$$\begin{aligned} [E_c(G)]^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i\neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]} \\ &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]} \\ &= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left|\prod_{i=1}^{n} \lambda_i\right|^{2/n} \\ &= 2m + |C| + n(n-1)D^{2/n}\ . \end{aligned}$$

Hence the result. □

Bapat and Pati showed that if the graph energy is a rational number, then it is an even integer [32] (see also [33]). The analogous result for minimum covering energy is:

Theorem 3.7 (Parity theorem)**.** *Let G be a graph with a minimum covering set C. If the minimum covering energy $E_c(G)$ of G is a rational number, then*

$$E_c(G) \equiv |C| \pmod{2} .$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be positive, and the rest of the minimum covering eigenvalues non-positive. Thus

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \cdots + \lambda_n)$$

implying

$$E_c(G) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - |C| .$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are algebraic integers, so is their sum. Hence $(\lambda_1 + \lambda_2 + \cdots + \lambda_r)$ must be an integer if $E_c(G)$ is rational. Hence the theorem. □

4 Minimum covering energies of some families of graphs

Theorem 4.1. *For $n \geq 2$, the minimum covering energy of a star graph $K_{1,n-1}$ is equal to $\sqrt{4n-3}$.*

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, center v_0, and the minimum covering set $C = \{v_0\}$. Then

$$A_c(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n} .$$

The characteristic polynomial of $A_c(K_{1,n-1})$ is

$$f_n(K_{1,n-1},\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda & 0 & \cdots & 0 & 0 \\ -1 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda & 0 \\ -1 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}_{n\times n}$$

$$= (-1)^{n+2} \begin{vmatrix} -1 & -1 & \cdots & -1 & -1 \\ \lambda & 0 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 \end{vmatrix} + (-1)^{2n}\,\lambda \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix} .$$

From this we get the recurrence relation

$$f_n(K_{1,n-1},\lambda) = -\lambda^{n-2} + \lambda\, f_{n-1}(K_{1,n-2},\lambda)\ . \tag{4.1}$$

Changing n to $(n-1)$ in (4.1), we obtain

$$f_{n-1}(K_{1,n-2},\lambda) = -\lambda^{n-3} + \lambda\, f_{n-2}(K_{1,n-3},\lambda)\ . \tag{4.2}$$

Combining (4.2) with (4.1), we deduce

$$f_n(K_{1,n-1},\lambda) = -2\lambda^{n-2} + \lambda^2\, f_{n-2}(K_{1,n-3},\lambda)\ .$$

Continuing this process, we find that

$$\begin{aligned} f_n(K_{1,n-1},\lambda) &= -(n-2)\lambda^{n-2} + \lambda^{n-2}\, f_2(K_{1,1},\lambda) \\ &= -(n-2)\lambda^{n-2} + \lambda^{n-2}\,(\lambda^2-\lambda-1) \\ &= \lambda^{n-2}\,[\lambda^2-\lambda-(n-1)]\ . \end{aligned}$$

Therefore, the minimum covering eigenvalues are $\frac{1}{2}\left(1+\sqrt{4n-3}\right)$, $\frac{1}{2}\left(1-\sqrt{4n-3}\right)$, and $0\ \ (n-2 \text{ times})$. Consequently, $E_c(K_{1,n-1}) = \sqrt{4n-3}$. □

Corollary 4.2. *Each positive integer* $2p-1$ (≥ 3) *is the minimum covering energy of a star graph.*

Proof. The minimum covering eigenvalues of K_{1,p^2-p} are $[p, 1-p, 0, 0, \ldots, 0]$. □

If the minimum covering eigenvalues are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and if their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then the *minimum covering spectrum* of the graph G will be written as

$$MC\ Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}$$

or

$$MC\ Spec(G) = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}) .$$

Theorem 4.3. *For $n \geq 2$, the minimum covering energy of the complete graph K_n is $\sqrt{(n+3)(n-1)}$.*

Proof. Let K_n be the complete graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and the minimum covering set $C = \{v_1, v_2, \cdots, v_{n-1}\}$. Then

$$A_c(K_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n\times n} .$$

The respective characteristic polynomial is

$$f_n(K_n, \lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda \end{vmatrix}_{n\times n}$$

$$= \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda-1 & -1 \\ 0 & 0 & 0 & \cdots & -\lambda & \lambda+1 \end{vmatrix}_{n\times n}$$

$$= \lambda \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & \lambda-1 & -1 \\ -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\lambda+1) \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 & -1 \\ -1 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & \lambda-1 & -1 \\ -1 & -1 & \cdots & -1 & \lambda-1 \end{vmatrix}$$

$$
\begin{aligned}
&= -\lambda^{n-1} + (\lambda+1)[\lambda^{n-2}(\lambda-(n-1))] \\
&= \lambda^{n-2}\left[\lambda^2-(n-1)\lambda-(n-1)\right] .
\end{aligned}
$$

Hence

$$
MC\ Spec(K_n) = \begin{pmatrix} 0 & \frac{n-1}{2} + \frac{\sqrt{(n+3)(n-1)}}{2} & \frac{n-1}{2} - \frac{\sqrt{(n+3)(n-1)}}{2} \\ n-2 & 1 & 1 \end{pmatrix}
$$

and the minimum covering energy of the complete graph is

$$
E_c(K_n) = \sqrt{(n+3)(n-1)} .
$$

□

Theorem 4.4. *The minimum covering energy of the complete bipartite graph $K_{m,n}$ is $(m-1)+\sqrt{4mn+1}$. In particular, if $n=m+1$ the energy is an integer $3m$.*

Proof. For the complete bipartite graph $K_{m,n}$ $(m \le n)$ with vertex set $V = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$, $C = \{u_1, u_2, \ldots, u_m\}$ is a minimum covering set. Then

$$
A_c(K_{m,n}) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{(m+n)\times(m+n)} .
$$

The characteristic polynomial of $A_c(K_{m,n})$ is

$$
\begin{aligned}
f_{m+n}(K_{m,n},\lambda) &= \begin{vmatrix}
\lambda-1 & 0 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\
0 & \lambda-1 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\
0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & -1 \\
-1 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & \lambda
\end{vmatrix} \\
&= \begin{vmatrix}
(\lambda-1)I_m & -J^T_{m\times n} \\
-J_{n\times m} & \lambda I_n
\end{vmatrix}
\end{aligned}
$$

where $J_{n\times m}$ is the matrix with all entries equal to unity. We have

$$\begin{aligned}
\begin{vmatrix} (\lambda-1)I_m & -J^T_{m\times n} \\ -J_{n\times m} & \lambda I_n \end{vmatrix} &= |(\lambda-1)I_m| \left|\lambda I_n - (-J)\frac{I_m}{\lambda-1}(-J^T)\right| \\
&= (\lambda-1)^{m-n}\left|\lambda(\lambda-1)I_n - J\,J^T\right| \\
&= (\lambda-1)^{m-n}\,P_{J\,J^T}[\lambda(\lambda-1)] \\
&= (\lambda-1)^{m-n}\,P_{m\,J_n}[\lambda(\lambda-1)]
\end{aligned}$$

where $P_{mJ_n}(\lambda)$ is the characteristic polynomial of the matrix $m\,J_n$. Therefore

$$\begin{aligned}
f_{m+n}(K_{m,n},\lambda) &= (\lambda-1)^{m-n}\,[\lambda(\lambda-1)-mn][\lambda(\lambda-1)]^{n-1} \\
&= (\lambda-1)^{m-1}\,\lambda^{n-1}\,[\lambda^2-\lambda-mn]
\end{aligned}$$

and

$$MC\ Spec(K_{m,n}) = \begin{pmatrix} 0 & 1 & \frac{1}{2}+\frac{\sqrt{4mn+1}}{2} & \frac{1}{2}-\frac{\sqrt{4mn+1}}{2} \\ n-1 & m-1 & 1 & 1 \end{pmatrix}.$$

Theorem 4.4 follows. □

The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i v_i \ : \ 1 \leq i, j \leq n\ ,\ i \neq j\}$. Therefore S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with the horizontal edges removed.

Theorem 4.5. *For $n \geq 3$, the minimum covering energy of the crown graph S_n^0 is equal to $(n-1)\sqrt{5} + \sqrt{4n-3}$.*

Proof. For the crown graph S_n^0 with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ we choose $C = \{u_1, u_2, \ldots, u_n\}$ as a minimum covering set. Then

$$A_c(S_n^0) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n}$$

and

$$\begin{aligned} f_{2n}(S_n^0,\lambda) &= \begin{vmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \lambda-1 & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda-1 & -1 & -1 & -1 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n} \\ &= \begin{vmatrix} (\lambda-1)I_n & -\mathbf{K}_n^T \\ -\mathbf{K}_n & \lambda I_n \end{vmatrix} \end{aligned}$$

where $\mathbf{K}_n$ is the ordinary adjacency matrix of the complete graph K_n . Observe that $\mathbf{K}_n^T = \mathbf{K}_n$ and that

$$\begin{aligned} f_{2n}(S_n^0,\lambda) &= \begin{vmatrix} (\lambda-1)I_n & -\mathbf{K}_n^T \\ -\mathbf{K}_n & \lambda I_n \end{vmatrix} \\ &= |(\lambda-1)I_n| \left| \lambda I_n - \left[-\mathbf{K}_n \frac{I_n}{\lambda-1}(-\mathbf{K}_n^T) \right] \right| \\ &= (\lambda-1)^n \left| \lambda\, I_n - \frac{\mathbf{K}_n\,\mathbf{K}_n^T}{\lambda-1} \right| \\ &= \left| \lambda(\lambda-1)I_n - \mathbf{K}_n^2 \right| = P_{\mathbf{K}_n^2}[\lambda(\lambda-1)] \end{aligned}$$

where $P_{\mathbf{K}_n^2}(\lambda)$ is the characteristic polynomial of the matrix $\mathbf{K}_n^2$. Therefore

$$\begin{aligned} f_{2n}(S_n^0,\lambda) &= [\lambda(\lambda-1)-1]^{n-1}\,[\lambda(\lambda-1)-(n-1)^2] \\ &= [\lambda^2-\lambda-1]^{n-1}\,[\lambda^2-\lambda-(n-1)^2]\ . \end{aligned}$$

Hence

$$MC\ Spec(S_n^0) = \begin{pmatrix} \frac{1}{2}+\frac{\sqrt{5}}{2} & \frac{1}{2}-\frac{\sqrt{5}}{2} & \frac{1}{2}+\frac{\sqrt{4n^2-8n+5}}{2} & \frac{1}{2}-\frac{\sqrt{4n^2-8n+5}}{2} \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$$

and the minimum covering energy of a crown graph is

$$E_c(S_n^0) = (n-1)\sqrt{5} + \sqrt{4n^2-8n+5}\ .$$

□

The cocktail party graph, denoted by $K_{n\times 2}$, is graph having vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \le i < j \le n\}$. This graph is also called as complete n-partite graph.

Theorem 4.6. *The minimum covering energy of the cocktail party graph $K_{n\times 2}$ is* $(2n-3) + \sqrt{4n^2+4n-7}$.

Proof. Let $K_{n\times 2}$ be the cocktail party graph with vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$ and let the minimum covering set be $C = \bigcup_{i=1}^{n-1}\{u_i, v_i\}$. Then

$$A_c(K_{n\times 2}) = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \end{pmatrix}_{2n\times 2n}.$$

The characteristic polynomial of $A_c(K_{n\times 2})$ is

$$f_{2n}(K_{n\times 2,\lambda}) = \begin{vmatrix} \lambda-1 & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 0 & \lambda-1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & \lambda-1 & 0 & \cdots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \lambda-1 & \cdots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \lambda-1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \lambda-1 & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & \lambda & 0 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \lambda \end{vmatrix}_{2n\times 2n}$$

$$= \begin{vmatrix} M & P^T \\ P & \lambda I_2 \end{vmatrix}_{2n\times 2n}$$

where

$$M = \begin{pmatrix} \lambda-1 & 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & \lambda-1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda-1 & 0 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \lambda-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \lambda-1 & 0 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \lambda-1 \end{pmatrix}_{(2n-2)\times(2n-2)}$$

and

$$P = \begin{pmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix}_{2\times(2n-2)} .$$

Note that

$$\begin{aligned} f_{2n}(K_{n\times 2,\lambda}) &= \begin{vmatrix} M & P^T \\ P & \lambda I_2 \end{vmatrix} = |M||\lambda I_2 - PM^{-1}P^T| \\ &= (\lambda-2n+3)(\lambda+1)^{n-2}\,(\lambda-1)^{n-1}\,|\lambda I_2 - PM^{-1}P^T| \\ &= (\lambda-2n+3)(\lambda+1)^{n-2}\,(\lambda-1)^{n-1} \begin{vmatrix} \lambda - \frac{(2n-2)}{\lambda-2n+3} & -\frac{(2n-2)}{\lambda-2n+3} \\ -\frac{(2n-2)}{\lambda-2n+3} & \lambda - \frac{(2n-2)}{\lambda-2n+3} \end{vmatrix} \\ &= \lambda(\lambda+1)^{n-2}\,(\lambda-1)^{n-1}\,[\lambda^2-(2n-3)\lambda-4(n-1)] \,. \end{aligned}$$

Therefore

$$MC\ Spec(K_{n\times 2}) = \begin{pmatrix} 0 & -1 & 1 & \frac{2n-3}{2}+\frac{\sqrt{4n^2+4n-7}}{2} & \frac{2n-3}{2}-\frac{\sqrt{4n^2+4n-7}}{2} \\ 1 & n-2 & n-1 & 1 & 1 \end{pmatrix}$$

and Theorem 4.6 follows. □

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