*Kragujevac J. Sci.* 33 (2011) 33–38. UDC 541.27:541.61

# ZAGREB POLYNOMIALS OF THORN GRAPHS

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*(Received May 5, 2010)*

ABSTRACT. We give the expressions for the two Zagreb polynomials of the thorn graphs, which are then generalized to general form.

## 1 Introduction

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, $d_G(u)$ or $d_u$ denotes the degree of $u$ in $G$. The first Zagreb index and the second Zagreb index, introduced in [1] and elaborated in [2], are defined respectively as:

$$ZG_1(G) = \sum_{u \in V(G)} d_u^2$$

$$ZG_2(G) = \sum_{uv \in E(G)} d_u d_v .$$

Note that $ZG_1(G)$ may be written as $ZG_1(G) = \sum_{uv \in E(G)} (d_u + d_v)$. The Zagreb indices reflect the extent of branching of the molecular carbon-atom skeleton, and can thus be viewed as molecular structure–descriptors [3,4]. Mathematical properties for these two graph invariants may be found in [5–12].

Recently, Fath–Tabar [13] put forward the first and the second Zagreb polynomials of the graph $G$, defined respectively as

$$ZG_1(G,x) = \sum_{uv\in E(G)} x^{d_u+d_v}$$

$$ZG_2(G,x) = \sum_{uv\in E(G)} x^{d_u d_v}$$

where $x$ is a dummy variable. For the graph $G$, we define

$$ZG^0_{p,q}(G,x) = \sum_{uv\in E(G)} x^{pd_u d_v+q(d_u+d_v)} \ .$$

Obviously, $ZG_1(G,x) = ZG^0_{0,1}(G,x)$ and $ZG_2(G,x) = ZG^0_{1,0}(G,x)$.

Another polynomial related to the first Zagreb index is

$$ZG^*_1(G,x) = \sum_{u\in V(G)} d_u x^{d_u}.$$

Define a polynomial as

$$ZG_0(G,x) = \sum_{u\in V(G)} x^{d_u}.$$

For a graph $G$ with vertex set $\{v_1,\ldots,v_n\}$, the thorn graph $G^*(p_1,\ldots,p_n)$ of $G$ is obtained by attaching $p_i$ thorns (pendent edges) to vertex $v_i \in V(G)$ for $i=1,\ldots,n$. We write $G^*_{a,b} = G^*(p_1,\ldots,p_n)$ if $p_i = ad_{v_i}+b$, where $a$ and $b$ are integers such that $p_i \geq 1$ for $v_i \in V(G)$. Let $V_1$ ($E_1$, respectively) be the set of pendent vertices (edges, respectively) of $G^*_{a,b}$ but not in $G$. Obviously, $V_1| = |E_1| = \sum_{i\in V(G)}(ad_i+b) = 2am+bn$. These kinds of thorn graphs are well known in chemistry [14].

In this paper we give the expressions for the two Zagreb polynomials of the thorn graphs, and then generalize them.

## 2 Zagreb polynomials of thorn graphs

**Theorem 1.** *Let $G$ be a graph with $n$ vertices and $m$ edges. Then*

$$\begin{aligned}
ZG_1(G^*_{a,b},x) &= x^{2b}ZG_1(G,x^{a+1}) + ax^{b+1}ZG_v(G,x^{a+1}) + bx^{b+1}ZG_0(G,x^{a+1})\\
ZG^*_1(G^*_{a,b},x) &= (a+1)x^b ZG_v(G,x^{a+1}) + bx^b ZG_0(G,x^{a+1}) + (2am+bn)x\\
ZG_2(G^*_{a,b},x) &= x^{b^2}ZG^0_{a+1,b}(G,x^{a+1}) + ax^b ZG_v(G,x^{a+1}) + bx^b ZG_0(G,x^{a+1}) \ .
\end{aligned}$$

**Proof.** For $u \in V(G^*_{a,b})$, $d^*_u$ denotes the degree of $u$ in $G^*_{a,b}$. If $u \in V(G) \subseteq V(G^*_{a,b})$, then $d^*_u = (a+1)d_u + b$.

For $ZG_1(G^*_{a,b}, x)$,

$$\begin{aligned}
ZG_1(G^*_{a,b}) &= \sum_{uv\in E(G)} x^{d^*_u+d^*_v} + \sum_{uv\in E_1} x^{d^*_u+d^*_v} \\
&= \sum_{uv\in E(G)} x^{[(a+1)d_u+b]+[(a+1)d_v+b]} \\
&+ \sum_{uv\in E_1, u\in V(G), v\in V_1} x^{(a+1)d_u+b+1} \\
&= \sum_{uv\in E(G)} x^{(a+1)(d_u+d_v)+2b} \\
&+ \sum_{u\in V(G)} (ad_u+b)x^{(a+1)d_u+b+1} \\
&= x^{2b} \sum_{uv\in E(G)} x^{(a+1)(d_u+d_v)} \\
&+ ax^{b+1} \sum_{u\in V(G)} d_u x^{(a+1)d_u} \\
&+ bx^{b+1} \sum_{u\in V(G)} x^{(a+1)d_u} \\
&= x^{2b} ZG_1(G, x^{a+1}) + ax^{b+1} ZG_v(G, x^{a+1}) \\
&+ bx^{b+1} ZG_0(G, x^{a+1}) \ .
\end{aligned}$$

For $ZG_v(G^*_{a,b}, x)$,

$$\begin{aligned}
ZG_v(G^*_{a,b}, x) &= \sum_{u\in V(G)} d^*_u x^{d^*_u} + \sum_{u\in V_1} 1 \cdot x^1 \\
&= \sum_{u\in V(G)} [(a+1)d_u+b] x^{(a+1)d_u+b} \\
&+ \sum_{u\in V(G)} (ad_u+b)x \\
&= (a+1)x^b \sum_{u\in V(G)} d_u x^{(a+1)d_u} \\
&+ bx^b \sum_{u\in V(G)} x^{(a+1)d_u} + (2am+bn)x \\
&= (a+1)x^b ZG_v(G, x^{a+1}) + bx^b ZG_0(G, x^{a+1}) \\
&+ (2am+bn)x \ .
\end{aligned}$$

For $ZG_2(G^*_{a,b},x)$,

$$\begin{aligned}
ZG_2(G^*_{a,b},x) &= \sum_{uv\in E(G)} x^{d^*_u d^*_v} + \sum_{uv\in E_1} x^{d^*_u d^*_v} \\
&= \sum_{uv\in E(G)} x^{[(a+1)d_u+b][(a+1)d_v+b]} \\
&+ \sum_{uv\in E_1, u\in V(G), v\in V_1} x^{[(a+1)d_v+b]\cdot 1} \\
&= \sum_{uv\in E(G)} x^{(a+1)^2 d_u d_v + b(a+1)(d_u+d_v)+b^2} \\
&+ \sum_{u\in V(G)} (ad_u+b) x^{(a+1)d_u+b} \\
&= x^{b^2} \sum_{uv\in E(G)} x^{(a+1)[(a+1)d_u d_v + b(d_u+d_v)]} \\
&+ ax^b \sum_{u\in V(G)} d_u x^{(a+1)d_u} + bx^b \sum_{u\in V(G)} x^{(a+1)d_u} \\
&= x^{b^2} ZG^0_{a+1,b}(G,x^{a+1}) + ax^b ZG_v(G,x^{a+1}) \\
&+ bx^b ZG_0(G,x^{a+1}) \ .
\end{aligned}$$

The proof is completed. ■

Similarly,

$$\begin{aligned}
ZG_0(G^*_{a,b},x) &= \sum_{u\in V(G)} x^{d^*_u} + \sum_{u\in V_1} x^1 \\
&= \sum_{u\in V(G)} x^{(a+1)d_u+b} + \sum_{u\in V(G)} (ad_u+b)x \\
&= x^b \sum_{u\in V(G)} x^{(a+1)d_u} + (2am+bn)x \\
&= x^b ZG_0(G,x^{a+1}) + (2am+bn)x \ .
\end{aligned}$$

**Theorem 2.** *Let $G$ be a graph with $n$ vertices and $m$ edges. Then*

$$\begin{aligned}
ZG^0_{p,q}(G^*_{a,b},x) &= x^{pb^2+2qb} ZG^0_{p(a+1),pb+q}(G,x^{a+1}) \\
&+ ax^{pb+q(b+1)} ZG_v(G,x^{(a+1)(p+q)}) \\
&+ bx^{pb+q(b+1)} ZG_0(G,x^{(a+1)(p+q)}) \ .
\end{aligned}$$

**Proof.** It is easily seen that

$$ZG^0_{p,q}(G^*_{a,b},x) = \sum_{uv\in E(G)} x^{pd^*_u d^*_v + q(d^*_u+d^*_v)} + \sum_{uv\in E_1} x^{pd^*_u d^*_v + q(d^*_u+d^*_v)}$$

$$
\begin{aligned}
&= \sum_{uv\in E(G)} x^{p[(a+1)^2 d_u d_v + b(a+1)(d_u+d_v)+b^2]+q[(a+1)(d_u+d_v)+2b]} \\
&+ \sum_{uv\in E_1} x^{p[(a+1)d_u+b]+q[(a+1)d_u+b+1]} \\
&= \sum_{uv\in E(G)} x^{p(a+1)^2 d_u d_v + (pb+q)(a+1)(d_u+d_v)+pb^2+2qb} \\
&+ \sum_{u\in V(G)} (ad_u+b) x^{(p+q)(a+1)d_u+pb+q(b+1)} \\
&= x^{pb^2+2qb} \sum_{uv\in E(G)} x^{(a+1)[p(a+1)d_u d_v + (pb+q)(d_u+d_v)]} \\
&+ x^{pb+q(b+1)} \sum_{u\in V(G)} (ad_u+b) x^{(p+q)(a+1)d_u} \\
&= x^{pb^2+2qb} ZG^0_{p(a+1),pb+q}(G, x^{a+1}) \\
&+ ax^{pb+q(b+1)} ZG_v(G, x^{(a+1)(p+q)}) \\
&+ bx^{pb+q(b+1)} ZG_0(G, x^{(a+1)(p+q)})
\end{aligned}
$$

as desired. ■

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