The (Weighted) Vertex $PI$ Index of Unicyclic Graphs

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(Received June 13, 2011)

Abstract

In [7], A. Ilić et al. introduced weighted vertex $PI$ index,

$$PI_w(G) = \sum_{e=uv \in E} (\deg(u) + \deg(v))(n_u(e) + n_v(e)),$$

where $\deg(u)$ denotes the vertex degree of $u$ and $n_u(e)$ denotes the number of vertices of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$. In this paper, we introduce three transformations to study the weighted vertex $PI$ index of the unicyclic graphs, and determine the smallest, second-smallest (for $n$ is odd), the largest $PI_w$ index of the unicyclic graphs and the extremal graphs. Also we show that the vertex $PI$ index of unicyclic graphs is equal to $n^2$ or $n^2 - n$.

1 Introduction

Let $G = (V, E)$ be a connected simple graph with vertex set $V(G)$ and edge set $E(G)$. For vertices $u, v \in V$, the distance $d(u, v)$ is defined as the length of the shortest path between $u$ and $v$ in $G$. The length of a path or a cycle is the number of its edges. A path or cycle of length $l$ is called a $l$–path or $l$–cycle. Other definitions not mentioned in this article can be found in [1].

Let $e = uv$ be an edge of the graph $G$, the number of vertices of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$ is denoted by $n_u(e)$. Analogously,
\( n_v(e) \) is the number of vertices of \( G \) whose distance to the vertex \( v \) is smaller than the distance to the vertex \( u \). The vertex \( PI \) index is defined as follows ([2, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17]):

\[
PI(G) = \sum_{e \in E} [n_u(e) + n_v(e)].
\]

One of the oldest degree-based graph invariants are the first Zagreb indices [3, 4, 12, 16, 18], defined as follows:

\[
M_1(G) = \sum_{u \in V(G)} \deg(u)^2,
\]

where \( \deg(u) \) denotes the vertex degree of \( u \). The vertex \( PI \) index, Zagreb indices and their variants have been used to study molecular complexity, chirality, in QSPR and QSAR analysis, etc([4, 13]).

In order to increase diversity for bipartite graphs, A. Ilić and N. Milosavljević ([10]) introduce weighted vertex of \( PI \) index as follows:

\[
PI_w(G) = \sum_{e=uv \in E} (\deg(u) + \deg(v))(n_u(e) + n_v(e)).
\]

For bipartite graphs it holds \( n_u(e) + n_v(e) = n \), and therefore the diversity of the original \( PI \) index is not satisfying. The inequality \( PI(G) \leq n \cdot m \) holds for a graph \( G \) with \( n \) vertices and \( m \) edges ([14, 15]), with equality holding if and only if \( G \) is bipartite.

This is why A. Ilić and N. Milosavljević introduced weighted version of \( PI \) index.

Assume that every edge \( e = uv \) has weight \( \deg(v) + \deg(u) \). Now, if \( G \) is a bipartite graph, then

\[
PI_w(G) = n \sum_{u \in V} \deg^2(u).
\]

This means that the weighted vertex \( PI \) index is directly connected to the first Zagreb index.

Let \( P_n \) and \( S_n \) denote the path and the star on \( n \) vertices, and let \( K_{p,q,r} \) denote the complete tripartite graph. In [10], A. Ilić et al. obtained sharp lower and upper bounds on \( PI_w \) index of connected graphs.

**Theorem 1.A.** ([10]) Let \( G \) be a connected graph on \( n \) vertices. Then

1. \( PI_w(G) \geq n(4n - 6) \), with equality holds if and only if \( G \cong P_n \).
2. \( PI_w(G) \leq \frac{8}{27}n^4 \), with equality holds if and only if \( 3|n \) and \( G \cong K_{\frac{2n}{3}, \frac{2n}{3}, \frac{n}{3}} \).

In this paper, we consider the weighted vertex \( PI \) index and vertex \( PI \) index of the unicyclic graphs. We introduce three transformations to study the weighted vertex \( PI \)
index of the unicyclic graphs in Section 2, determine the smallest, second-smallest (for \( n \) is odd) \( PI_w \) index of the unicyclic graphs and the extremal graphs in Section 3, determine the largest \( PI_w \) index of the unicyclic graphs and the extremal graphs in Section 4, and we show the \( PI \) index of the unicyclic graphs is a constant in Section 5.

2 Transformations

In this section, we introduce three transformations which are important to our main results.

A rooted graph has one of its vertex, called the root, distinguished from the others. Let \( T_1, T_2, \ldots, T_k \) be \( k \) rooted trees with \( |V(T_i)| \geq 2 \) \((1 \leq i \leq k)\) and roots \( u_{11}, u_{21}, \ldots, u_{k1} \), respectively. Let \( C_r \) be a cycle with length \( r \) \((r \geq 3)\).

Define \( G(n, r, 0) = C_n \). For \( 1 \leq k \leq r \leq n \), define \( G(n, r, k) \) be a unicyclic graph on \( n \) vertices obtained from \( C_r, T_1, T_2, \ldots, T_k \), by attaching \( k \) rooted trees \( T_1, T_2, \ldots, T_k \) to \( k \) distinct vertices of the cycle \( C_r \), that is to say, \( G(n, r, k) \) is a unicyclic graph on \( n \) vertices by identifying some vertex of \( C_r \) with the root \( u_{i1} \) of \( T_i \) for each \( i \) \((1 \leq i \leq k)\). Let \( m_i \) be the number of vertices of \( T_i \) for \( 1 \leq i \leq k \), then \( m_i \geq 2 \) and \( \sum_{i=1}^{k} m_i = n - r + k \).

Let \( \mathbb{P}^* = \{ P | P \) is a rooted path and the root is its starting vertex\} and \( \mathbb{S}^* = \{ S | S \) is a rooted star and the root is its center\}. For a path \( P \in \mathbb{P}^* \) and a star \( S \), the rooted graph \( P + S \) is obtained by identifying the end vertex of \( P \) with the center of \( S \) and the root of \( P + S \) is the root of \( P \). Define \( \mathbb{PS}^* = \{ P + S | P \in \mathbb{P}^* \) and \( S \) is a star\}.

**Proposition 2.1.** Let \( G \) be a connected unicyclic graph and the length of the unique cycle \( C_r \) is \( r \),

1. If \( r \) is even, then \( n_u(e) + n_v(e) = n \) for any \( e = uv \in E(G) \).
2. If \( r \) is odd, then \( n_u(e) + n_v(e) = n \) for any \( e = uv \in E(G\setminus C_r) \); and for each edge \( e = uv \in E(C_r) \), \( n_u(e) + n_v(e) = n - a_e \), where \( a_e \) \((1 \leq a_e \leq n - 2)\) is the number of vertices that are equidistant from vertex \( u \) and vertex \( v \).

2.1 \( \alpha - \) transformation

Let \( n, r, k \) be integers and \( 1 \leq k \leq r \leq n \). Let \( \mathcal{G}_1(n, r, k) \) be the set of all unicyclic graphs on \( n \) vertices that obtained from \( C_r \) by attaching \( k \) rooted paths in \( \mathbb{P}^* \) to \( k \) distinct vertices of \( C_r \), see Fig.1.
Fig. 1. A graph $G_1(n, r, k)$ in the set $\mathbb{G}_1(n, r, k)$

$\alpha$-transformation: Let $G(n, r, k)$ be defined as above, if $G(n, r, k) \notin \mathbb{G}_1(n, r, k)$, then there exists at least a rooted tree $T_i \notin \mathbb{P}^*$ for some $i \in \{1, \ldots, k\}$. Let the longest path starting from $u_{i1}$ of $T_i$ be $u_{i1}u_{i2}\cdots u_{il}$, and $y(\neq u_{il})$ be any pendant vertex of $T_i$ which adjacent to vertex $x$, $G'(n, r, k)$ is obtained from $G(n, r, k)$ by deleting a pendant edge $xy$ and adding a pendant edge $u_{il}y$.

**Lemma 2.1.** Let $G'(n, r, k)$ be the graph obtained from $G(n, r, k)$ by $\alpha$-transformation, then $PI_w(G(n, r, k)) \geq PI_w(G'(n, r, k))$.

**Proof.** For convenience, let $G = G(n, r, k), G' = G'(n, r, k)$. We only need to consider the edges whose weight have been changed by $\alpha$-transformation. Note that only the degree of vertices $x$ and $u_{il}$ have been changed and we consider the following two cases.

**Case 1:** $x \neq u_{i1}$.

In this case, $x \notin V(C_r)$ and $\text{deg}(x) \geq 2$, so the edges which are adjacent to vertex $x$ or $u_{il}$ don’t belong to $E(C_r)$, thus by the definition of $PI_w$ index and Proposition 2.1, we consider the two subcases.

**Subcase 1.1:** $x \neq u_{i,l-1}$ (see Fig.2). The changes of the degrees are listed in Table 1. Then

$$PI_w(G) - PI_w(G') = n \cdot \left[ \sum_{xv \in E(G)} \left( \text{deg}(x) + \text{deg}(v) \right) + \left( \text{deg}(u_{i,l-1}) + \text{deg}(u_{i,l}) \right) \right]$$

$$- n \cdot \left\{ \sum_{xv \in E(G')} \left[ \left( \text{deg}(x) - 1 \right) + \text{deg}(v) \right] + \left( \text{deg}(u_{i,l-1}) + 2 \right) + (2 + 1) \right\}$$

$$= n \cdot \left[ \text{deg}(x) + \text{deg}(y) + \sum_{xv \in E(G')} 1 - 4 \right] = 2n(\text{deg}(x) - 2) \geq 0.$$
Fig. 2. \( \alpha \)-transformation for \( x \neq u_{i1}, u_{i,l-1} \)

Table 1.

<table>
<thead>
<tr>
<th>vertex</th>
<th>( x )</th>
<th>( y )</th>
<th>( u_{i,l-1} )</th>
<th>( u_{il} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree of ( G )</td>
<td>( \deg(x) )</td>
<td>( \deg(y) = 1 )</td>
<td>( \deg(u_{i,l-1}) )</td>
<td>( \deg(u_{il}) = 1 )</td>
</tr>
<tr>
<td>degree of ( G' )</td>
<td>( \deg(x) - 1 )</td>
<td>1</td>
<td>( \deg(u_{i,l-1}) )</td>
<td>2</td>
</tr>
</tbody>
</table>

**Subcase 1.2:** \( x = u_{i,l-1} \) (see Fig. 3). In this case, \( \deg(x) \geq 3 \), the changes of the degrees are listed in Table 2. Then

\[
P I_w(G) - P I_w(G') = n \left[ \sum_{x \in E(G)} (\deg(x) + \deg(v)) \right] \\
- n \left[ \sum_{x \in E(G'), x \neq u_{il}} ((\deg(x) - 1) + \deg(v)) \right] \\
- n \cdot \left[ ((\deg(x) - 1) + 2) + (2 + 1) \right] \\
= n \left[ \deg(x) + \deg(y) + \deg(u_{i,l}) + \sum_{x \in E(G'), x \neq u_{i,l}} 1 - 4 \right] \\
= 2n(\deg(x) - 2) > 0 .
\]

Fig. 3. \( \alpha \)-transformation for \( x \neq u_{i1} \) and \( x = u_{i,l-1} \)

Table 2.

<table>
<thead>
<tr>
<th>vertex</th>
<th>( x = u_{i,l-1} )</th>
<th>( y )</th>
<th>( u_{il} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree of ( G )</td>
<td>( \deg(x) )</td>
<td>( \deg(y) = 1 )</td>
<td>( \deg(u_{il}) = 1 )</td>
</tr>
<tr>
<td>degree of ( G' )</td>
<td>( \deg(x) - 1 )</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

**Case 2:** \( x = u_{i1} \).

In this case, \( x \in V(C_r) \) and \( \deg(x) \geq 4 \), let vertices \( y_1, y_2 \in V(C_r) \) such that \( xy_1, xy_2 \in E(C_r) \), then there exist integers \( a, b (0 \leq a, b \leq n - 2) \) such that \( n_x(xy_1) + n_y(xy_1) = n - a \)
and $n_x(xy_2) + n_{y_2}(xy_2) = n - b$ by Proposition 2.1, the other edges which are adjacent to vertex $x$ or $u_i$ don’t belong to $E(C_r)$, thus by the definition of $PI_w$ index and Proposition 2.1, we consider two subcases.

**Subcase 2.1:** $x \neq u_{i,l-1}$ (see Fig.4). The changes of the degrees are listed in Table 1. Then

\[
PI_w(G) - PI_w(G') = (n - a)(\text{deg}(x) + \text{deg}(y_1)) + (n - b)(\text{deg}(x) + \text{deg}(y_2))
\]

\[
+ n \cdot \left[ \sum_{xv \in E(G), v \neq y_1, y_2} (\text{deg}(x) + \text{deg}(v)) + (\text{deg}(u_{i,l-1}) + \text{deg}(u_{il})) \right]
\]

\[
- (n - a)[(\text{deg}(x) - 1) + \text{deg}(y_1)] - (n - b)[(\text{deg}(x) - 1) + \text{deg}(y_2)]
\]

\[
- n \cdot \left\{ \sum_{xv \in E(G'), v \neq y_1, y_2} [(\text{deg}(x) - 1) + \text{deg}(v)] \right\}
\]

\[
- n \cdot [(\text{deg}(u_{i,l-1}) + 2) + (2 + 1)]
\]

\[
= (n - a) + (n - b) + n \cdot \left[ \text{deg}(x) + \text{deg}(y) + \sum_{xv \in E(G'), v \neq y_1, y_2} 1 - 4 \right]
\]

\[
= 2n - a - b + 2n(\text{deg}(x) - 3) > 0 .
\]

![Fig.4](image1.png)

**Subcase 2.2:** $x = u_{i,l-1}$ (see Fig.5). The changes of degrees are listed in Table 2.
Then
\[ PI_w(G) - PI_w(G') = (n-a)(\text{deg}(x) + \text{deg}(y_1)) + (n-b)(\text{deg}(x) + \text{deg}(y_2)) \]
\[ + n \cdot \left( \sum_{xv \in E(G), v \neq y_1, y_2} (\text{deg}(x) + \text{deg}(v)) \right) \]
\[ - (n-a)[(\text{deg}(x) - 1) + \text{deg}(y_1)] - (n-b)[(\text{deg}(x) - 1) + \text{deg}(y_2)] \]
\[ - n \cdot \left\{ \sum_{xv \in E(G'), v \neq y_1, y_2, u_{i,l}} [(\text{deg}(x) - 1) + \text{deg}(v)] \right\} \]
\[ - n \cdot [(\text{deg}(x) - 1) + 2 + (2 + 1)] \]
\[ = (n-a) + (n-b) \]
\[ + n \cdot \left[ \text{deg}(x) + \text{deg}(y) + \text{deg}(u_{i,l}) + \sum_{xv \in E(G'), v \neq y_1, y_2, u_{i,l}} 1 - 4 \right] \]
\[ = 2n - a - b + 2n(\text{deg}(x) - 3) > 0. \]

Combing the above arguments, we obtain the result. \( \square \)

By the proof of Lemma 2.1, we have \( PI_w(G(n, r, k)) > PI_w(G'(n, r, k)) \) in Subcase 1.2 and Case 2. Thus we have \( \text{deg}(x) = 2 \) for every \( \alpha \)-transformation if \( PI_w(G(n, r, k)) = PI_w(G'(n, r, k)) \), and it is a contradiction if \( G(n, r, k) \not\cong G'(n, r, k) \) because there must exist some vertex \( x \) (say, \( x = u_{ij} \) for \( j \in \{2, 3, \ldots, l-1\} \)) satisfying \( \text{deg}(x) = 3 \). Thus we have

**Lemma 2.2.** Let \( G(n, r, k) \) be defined as above, and \( G'(n, r, k) \) obtained from \( G(n, r, k) \) by repeating \( \alpha \)-transformation, and we cannot get other graph from \( G'(n, r, k) \) by repeating \( \alpha \)-transformation, then

1. \( G'(n, r, k) \in \mathbb{G}_1(n, r, k) \).
2. \( PI_w(G(n, r, k)) \geq PI_w(G'(n, r, k)) \), and the equality holds if and only if \( G(n, r, k) \cong G'(n, r, k) \).

### 2.2 \( \beta \)-transformation

Let \( n, r, k \) be integers and \( 1 \leq k \leq r \leq n \). Let \( \mathbb{G}_2(n, r, k) \) be the set of all unicyclic graphs on \( n \) vertices that obtained from \( C_r \) by attaching \( k \) rooted stars in \( S^* \) to \( k \) distinct vertices of \( C_r \) (see Fig.6).
Let $PS_i = P_i + S_i \in PS^*$ be defined as above, and $|V(P_i)| = p_i$, $|V(S_i)| = m_i - p_i + 1$, $|V(PS_i)| = m_i$ for each $i \in \{1, 2, \ldots, k\}$. If the rooted tree $T_i \in S^*$ or $T_i \in PS^*$ for each $i \in \{1, 2, \ldots, k\}$, we define $G_{3}(n, r, k)$ be the set of all unicyclic graphs on $n$ vertices that obtained from $C_r$ by attaching $k$ rooted trees $T_1, T_2, \ldots, T_k$ to $k$ distinct vertices of $C_r$ (see Fig.7 and Fig.8). Clearly, $G_{2}(n, r, k) \subset G_{3}(n, r, k)$.

Fig.6. A graph $G_2(n, r, k)$ in the set $G_{2}(n, r, k)$

Fig.7. A graph $G_3(n, r, k)$ in the set $G_{3}(n, r, k)$

Fig.8. A graph $G_4(n, r, k)$ in the set $G_{3}(n, r, k)$

**β-transformation:** Let $G(n, r, k)$ be defined as above, if $G(n, r, k) \not\in G_{3}(n, r, k)$, then there exists at least a rooted tree $T_i \notin S^*$ and $T_i \notin PS^*$ for some $i \in \{1, \ldots, k\}$. Let $x$ be one of the maximum degree vertices of $T_i$ and $z$ be an any pendent vertex of $T_i$ which adjacent to vertex $y (y \neq x)$. $G'(n, r, k)$ is obtained from $G(n, r, k)$ by deleting a pendent edge $yz$ and adding a pendent edge $xz$.

**Lemma 2.3.** Let $G'(n, r, k)$ be obtained from $G(n, r, k)$ by β-transformation, then

$$PI_w(G(n, r, k)) < PI_w(G'(n, r, k)).$$

**Proof.** For convenience, let $G = G(n, r, k), G' = G'(n, r, k)$. We only need to consider the edges whose weight have been changed by β-transformation. Note that only the degree of vertices $x$ and $u_{it}$ have been changed and we consider the following three cases.

**Case 1:** $x \neq u_{i1}$ and $y \neq u_{i1}$ (see Fig.9).
In this case, \( x \notin V(C_r), \ y \notin V(C_r) \) and \( \deg(x) \geq \deg(y) \), Hence the edges which adjacent to vertex \( x \) or \( y \) don’t belong to \( E(C_r) \). By the definition of \( PI_w \) index and Proposition 2.1, we have

\[
PI_w(G) - PI_w(G') = n \cdot \left[ \sum_{xv \in E(G)} (\deg(x) + \deg(v)) + \sum_{yw \in E(G)} (\deg(y) + \deg(w)) \right]
- n \cdot \left[ \sum_{xv \in E(G')} ((\deg(x) + 1) + \deg(v)) \right]
- n \cdot \left[ \sum_{yw \in E(G')} ((\deg(y) - 1) + \deg(w)) \right]
= n \cdot \left[ \sum_{xv \in E(G')} (-1) - \deg(x) - 1 + \sum_{yw \in E(G')} 1 + \deg(y) + 1 \right]
= 2n(\deg(y) - \deg(x) - 1) < 0 .
\]

**Case 2:** \( x = u_{i_1} \) (see Fig.10).

In this case, \( x \in V(C_r), \ y \notin V(C_r) \) and \( \deg(x) \geq \deg(y) \). Let vertices \( y_1, y_2 \in V(C_r) \) such that \( xy_1, xy_2 \in E(C_r) \), then there exist integers \( a, b (0 \leq a, b \leq n - 2) \) such that \( n_x(xy_1) + n_{y_1}(xy_1) = n - a \) and \( n_x(xy_2) + n_{y_2}(xy_2) = n - b \) by Proposition 2.1, and the other edges which adjacent to vertex \( x \) or \( y \) don’t belong to \( E(C_r) \). Thus by the definition
of $PI_w$ index and Proposition 2.1, we have

$$PI_w(G) - PI_w(G') = (n - a)(\deg(x) + \deg(y_1)) + (n - b)(\deg(x) + \deg(y_2))$$

$$+ n \cdot \left[ \sum_{xv \in E(G), v \neq y_1, y_2} (\deg(x) + \deg(v)) + \sum_{yw \in E(G)} (\deg(y) + \deg(w)) \right]$$

$$- (n - a)[(\deg(x) + 1) + \deg(y_1)] - (n - b)[(\deg(x) + 1) + \deg(y_2)]$$

$$- n \cdot \left[ \sum_{xv \in E(G'), v \neq y_1, y_2} ((\deg(x) + 1) + \deg(v)) \right]$$

$$- n \cdot \sum_{yw \in E(G')} ((\deg(y) - 1) + \deg(w))$$

$$= -(n - a) - (n - b)$$

$$+ n \cdot \left[ \sum_{xv \in E(G'), v \neq y_1, y_2} (-1) - \deg(x) - 1 + \sum_{yw \in E(G')} 1 + \deg(y) + 1 \right]$$

$$= -(2n - a - b) + 2n(\deg(y) - \deg(x)) < 0 .$$

Case 3: $y = u_i$ (see Fig.11).

In this case, $x \not\in V(C_r)$, $y \in V(C_r)$ and $\deg(x) \geq \deg(y)$. Let vertices $w_1, w_2 \in V(C_r)$ such that $y w_1, y w_2 \in E(C_r)$, then there exist integers $a, b (0 \leq a, b \leq n - 2)$ such that $n_y(y w_1) + n_w(y w_1) = n - a$ and $n_y(y w_2) + n_w(y w_2) = n - b$ by Proposition 2.1, and the other edges which adjacent to vertex $x$ or $y$ are not belong to $E(C_r)$. Thus by the
The definition of $PI_w$ index and Proposition 2.1, we have

$$PI_w(G) - PI_w(G') = n \cdot \left[ \sum_{xv \in E(G)} (\deg(x) + \deg(v)) \right]$$

$$+ n \cdot \left[ \sum_{yw \in E(G), w \neq w_1, w_2} (\deg(y) + \deg(w)) \right]$$

$$+ (n - a)(\deg(y) + \deg(w_1)) + (n - b)(\deg(y) + \deg(w_2))$$

$$- (n - a)((\deg(y) - 1) + \deg(w_1)) - (n - b)((\deg(y) - 1) + \deg(w_2))$$

$$- n \cdot \left[ \sum_{xv \in E(G')} ((\deg(x) + 1) + \deg(v)) \right]$$

$$- n \cdot \left[ \sum_{yw \in E(G'), w \neq w_1, w_2} ((\deg(y) - 1) + \deg(w)) \right]$$

$$= n \cdot \left[ \sum_{xv \in E(G')} (-1) - \deg(x) - 1 \right]$$

$$+ n \cdot \left[ \sum_{yw \in E(G'), w \neq w_1, w_2} 1 + \deg(y) + 1 \right] + (n - a) + (n - b)$$

$$= 2n(\deg(y) - \deg(x) - 1) - a - b < 0.$$

Combining the above three cases, we obtain the result. □

**Lemma 2.4.** Let $G(n, r, k)$ be defined as above, $G'(n, r, k)$ obtained from $G(n, r, k)$ by repeating $\beta$-transformation, and we cannot get other graph from $G'(n, r, k)$ by repeating $\beta$-transformation, then $G'(n, r, k) \in \mathbb{G}_3(n, r, k)$ and $PI_w(G(n, r, k)) < PI_w(G'(n, r, k)).$

### 2.3 $\gamma$-transformation

**$\gamma$-transformation:** Let $G(n, r, k)$ be defined as above. If there exists at least a rooted tree $T_i = P_i + S_i \in P^*_S$ for some $i \in \{1, 2, \ldots, k\}$, for convenience, let $|V(T_i)| = s + t$, $V(P_i) = \{u_1, u_2, \ldots, u_s\}$, $V(S_i) = \{w_1, w_2, \ldots, w_t\}$, and $E(T_i) = \{u_1u_2, u_2u_3, \ldots, u_{s-1}u_s, u_sw_1, u_sw_2, \ldots, u_sw_t\}$. $G'(n, r, k)$ is obtained from $G(n, r, k)$ by deleting the pendent edges $u_sw_j$ and adding a pendent edges $u_1w_j$ for all $j \in \{1, 2, \ldots, t\}$ (see Fig. 12).
Lemma 2.5. Let $G'(n, r, k)$ be obtained from $G(n, r, k)$ by $\gamma$-transformation, then

$$PI_w(G(n, r, k)) < PI_w(G'(n, r, k)).$$

![Fig.12. $\gamma$-transformation](image)

Proof. For convenience, let $G = G(n, r, k), G' = G'(n, r, k)$. We only need to consider the edges whose weight have been changed by $\gamma$-transformation. Note that only the degree of vertices $u_1$ and $u_s$ have been changed, $u_1 \in V(C_r), u_s \not\in V(C_r),$ and $deg(u_1) = 3$.

Let vertices $y_1, y_2 \in V(C_r)$ such that $u_1y_1, u_1y_2 \in E(C_r)$, and the other edges which adjacent to vertex $u_1$ or $u_s$ aren’t in $E(C_r)$. Therefore there exist integers $a, b(0 \leq a, b \leq n - 2)$ such that $n_{u_1}u_1y_1 + n_{y_1}u_1y_1 = n - a$ and $n_{u_1}u_1y_2 + n_{y_2}u_1y_2 = n - b$ by Proposition 2.1.

By the definition of $PI_w$ index and Proposition 2.1, we have

**Case 1:** $u_s \neq u_2$.

$$PI_w(G) - PI_w(G') = (n - a)(deg(u_1) + deg(y_1)) + (n - b)(deg(u_1) + deg(y_2))$$

$$+ n \cdot \left[ \sum_{u_1v \in E(G), v \neq y_1, y_2} (deg(u_1) + deg(v)) + \sum_{u_s v \in E(G)} (deg(u_s) + deg(v)) \right]$$

$$- (n - a)((deg(u_1) + t) + deg(y_1)) - (n - b)((deg(u_1) + t) + deg(y_2))$$

$$- n \cdot \left[ \sum_{u_1v \in E(G'), v \neq y_1, y_2} ((deg(u_1) + t) + deg(v)) \right]$$

$$- n \cdot [(deg(u_s) - t) + deg(u_{s-1})]$$

$$= -t(n - a) - t(n - b) + n \cdot \left[ \sum_{u_1v \in E(G'), v \neq y_1, y_2} (-t) - tdeg(u_1) - t \right]$$

$$+ n \cdot [(t + 1)(t + 1) + deg(u_{s-1}) + t - (1 + deg(u_{s-1}))]$$

$$= -t(2n - a - b) - 2tn(deg(u_1) - 2) = -t(4n - a - b) < 0.$$
Case 2: \( u_s = u_2 \).

\[
PI_w(G) - PI_w(G') = (n - a)(\deg(u_1) + \deg(y_1)) + (n - b)(\deg(u_1) + \deg(y_2))
\]

\[
+ n \cdot (\deg(u_1) + \deg(u_2)) + n \cdot \left[ \sum_{u_1 \in E(G), u \neq y_1, y_2, u_2} (\deg(u_1) + \deg(v)) \right]
\]

\[
+ n \cdot \left[ \sum_{u_2 \in E(G), u \neq u_1} (\deg(u_2) + \deg(w)) \right]
\]

\[
- (n - a)((\deg(u_1) + t) + \deg(y_1)) - (n - b)((\deg(u_1) + t) + \deg(y_2))
\]

\[
- n \cdot [(\deg(u_1) + t) + (\deg(u_2) - t)]
\]

\[
- n \cdot \left[ \sum_{u_1 \in E(G'), u \neq y_1, y_2, u_2} ((\deg(u_1) + t) + \deg(v)) \right]
\]

\[
= -t(n - a) - t(n - b)
\]

\[
+ n \cdot \left[ \sum_{u_1 \in E(G'), u \neq y_1, y_2, u_2} (-t) - t\deg(u_1) - t + (t + 1)t + t \right]
\]

\[
= -t(2n - a - b) - 2tn(\deg(u_1) - 2) = -t(4n - a - b) < 0.
\]

Combining the above two cases, we obtain the result. \(\square\)

**Remark.** Note that if \( \deg(u_1) \geq 3 \), then the result of Lemma 2.5 still holds, and the \( \gamma \)-transformation can increase the weighted vertex \( PI \) index for more graphs.

**Lemma 2.6.** Let \( G(n, r, k) \) be defined as above, \( G'(n, r, k) \) obtained from \( G(n, r, k) \) by repeating \( \beta \)-transformation and \( \gamma \)-transformation, and we cannot get other graph from \( G'(n, r, k) \) by repeating the above two transformations, then \( G'(n, r, k) \in G_2(n, r, k) \) and \( PI_w(G(n, r, k)) < PI_w(G'(n, r, k)) \).

### 3 The lower bounds of \( PI_w \) index of unicyclic graphs

In this section, the smallest, second-smallest (only for \( n \) is odd) \( PI_w \) index of unicyclic graphs and the extremal graphs will be obtained.

**Proposition 3.1.** Let \( n \geq 3 \), then

\[
PI_w(C_n) = \begin{cases} 4n^2, & \text{if } n \text{ is even}, \\ 4n(n - 1), & \text{if } n \text{ is odd}. \end{cases}
\] (3.1)
Proof. Case 1: If \( n \) is even, then for any edge \( e = uv \in E(C_n) \), \( n_u(e) + n_v(e) = n \), and the degree of any vertex is 2. Hence \( PI_w(C_n) = 4n^2 \).

Case 2: If \( n \) is odd, then for any edge \( e = uv \in E(C_n) \), \( n_u(e) + n_v(e) = n - 1 \), and the degree of any vertex is 2. Hence \( PI_w(C_n) = 4n(n - 1) \). □

Let \( G(n, r, k), T_i, m_i = |V(T_i)| \) be defined as above, \( V(C_r) = \{v_1, v_2, \ldots, v_r\} \), for convenience, we introduce a new notation \( l_j \) for each \( j (1 \leq j \leq r) \) as follows:

\[
l_j = \begin{cases} m_i, & \text{if vertex } v_j \text{ is the root } u_{i_1} \text{ of } T_i \text{ for some } i (1 \leq i \leq k); \\ 1, & \text{otherwise.} \end{cases}
\]

Clearly, \( \sum_{j=1}^{r} l_j = n \). By the notation of \( l_j \), we have

Proposition 3.2. Let \( G(n, r, k) \) be defined as above, and the length \( r \) of cycle \( C_r \) be odd, \( V(C_r) = \{v_1, v_2, \ldots, v_r\} \), then for any edge \( e = v_{i}v_{i+1} \in E(C_r) \),

1. \( n_{v_i}(e) + n_{v_{i+1}}(e) = n - l_i + \frac{i}{r+1} \), where the subscripts \( i, i + 1, i + \frac{i}{r+1} \) are modular \( r \).

2. If \( G(n, r, k) \in G_1(n, r, k) \), then the weight \( \deg(v_i) + \deg(v_{i+1}) = 6 - \lfloor \frac{i}{r} \rfloor - \lfloor \frac{i}{r+1} \rfloor \), where \( \lfloor x \rfloor \) is the greatest integer not exceeding \( x \).

3. If \( G(n, r, k) \in G_2(n, r, k) \), then the weight \( \deg(v_i) + \deg(v_{i+1}) = l_i + l_{i+1} + 2 \).

Proof. It is obvious that (1) and (3) hold. We only prove (2).

For any edge \( e = v_{i}v_{i+1} \in E(C_r) \), we have

\[
\deg(v_i) + \deg(v_{i+1}) = \begin{cases} 4, & \text{if } l_i = 1, l_{i+1} = 1; \\ 5, & \text{if } l_i = 1, l_{i+1} > 1 \text{ or } l_i > 1, l_{i+1} = 1; \\ 6, & \text{if } l_i > 1, l_{i+1} > 1. \end{cases}
\]

Then \( \deg(v_i) + \deg(v_{i+1}) = 6 - \lfloor \frac{i}{r} \rfloor - \lfloor \frac{i}{r+1} \rfloor \). □

Theorem 3.1. Let \( n, r \) be integers with \( 3 \leq r \leq n \), \( G \cong C_n \) be a connected unicyclic graph on \( n \) vertices, and the length of unique cycle of \( G \) be \( r \), then

\[
PI_w(G) \geq \begin{cases} 4n^2 + 2n, & \text{if } r \text{ is even}, \\ 4n^2 - 2n - 2, & \text{if } r \text{ is odd}, \end{cases}
\]

with the equality holds if and only if \( G \in G_1(n, r, 1) \).

Proof. There exists an integer \( k (1 \leq k \leq r \leq n) \) such that \( G = G(n, r, k) \) by \( G \not\cong C_n \).

Suppose \( G_1(n, r, k) \in G_1(n, r, k) \) is obtained from \( G(n, r, k) \) by repeating \( \alpha - \) transformation. By Lemma 2.2, \( PI_w(G(n, r, k)) \geq PI_w(G_1(n, r, k)) \). Thus we only need to compute the value of \( PI_w(G_1(n, r, k)) \) for \( k \geq 1 \). For convenience, take \( G_1 = G_1(n, r, k) \).

Case 1: \( k = 1 \).
Subcase 1.1: $r$ is even.

For any edge $e = uv \in E(G_1)$, then $n_u(e) + n_v(e) = n$, and one vertex has degree 3, one vertex has degree 1 and $n - 2$ vertices have degree 2. By the definition of $PI_w$ index, we have

$$PI_w(G_1) = n \cdot \left[ \sum_{uv \in E(G_1)} (\deg(u) + \deg(v)) \right] = n \cdot \left[ \sum_{u \in V(G_1)} \deg(u)^2 \right]$$

$$= n \cdot (3^2 + 1^2 + 2^2(n - 2)) = 4n^2 + 2n .$$

Subcase 1.2: $r$ is odd.

For any edge $e = uv \in E(G_1 \setminus C_n)$, then $n_u(e) + n_v(e) = n$. Note that $m_1 = n - r + 1$, by Proposition 3.2, there is only one edge $e = uv \in E(C_r)$ satisfying $n_u(e) + n_v(e) = n - (n - r + 1) = r - 1$ whose weight is $2 + 2 = 4$, and there are $r - 1$ edges $e = xy \in E(C_r)$ satisfying $n_x(e) + n_y(e) = n - 1$, where 2 edges have weight $2 + 3 = 5$, and other $r - 3$ edges have weight $2 + 2 = 4$. By the definition of $PI_w$ index, we have

$$PI_w(G_1) = 4(r-1) + 4(r-3)(n-1) + 2 \cdot 5(n-1) + n \cdot [(2+3) + (2+1) + (n-r-2)(2+2)]$$

$$= 4n^2 - 2n - 2.$$

Case 2: $k \geq 2$.

Subcase 2.1: If $r$ is even, then for any edge $e = uv \in E(G_1)$, $n_u(e) + n_v(e) = n$, and $k$ vertices have degree 3, $k$ vertices have degree 1, $n - 2k$ vertices have degree 2. By the definition of $PI_w$ index, we have

$$PI_w(G_1) = n \cdot \left[ \sum_{uv \in E(G_1)} (\deg(u) + \deg(v)) \right] = n \cdot \left[ \sum_{u \in V(G_1)} \deg(u)^2 \right]$$

$$= n \cdot (3^2k + 1^2k + 2^2(n - 2k)) = 4n^2 + 2kn > 4n^2 + 2n .$$

Subcase 2.2: If $r$ is odd, for any edge $e = uv \in E(G_1 \setminus C_n)$, then $n_u(e) + n_v(e) = n$. By Proposition 3.2 and the definition of $PI_w$ index, we have
Combining the above two cases, we complete the proof.

From Proposition 3.1 and Theorem 3.1, we have

**Theorem 3.2.** Let $n, r$ be integers with $3 \leq r \leq n$, $G$ be a connected unicyclic graph on $n$ vertices, and the length of unique cycle of $G$ be $r$, then

\[
PI_w(G) \geq \begin{cases} 
4n^2 - 4n, & \text{if } n \text{ is odd,} \\
4n^2 - 2n - 2, & \text{if } n \text{ is even,}
\end{cases}
\]

with the equality holds if and only if $G \in \mathbb{G}_1(n, r, 1)$ (r is odd) when $n$ is even and $G \cong C_n$ when $n$ is odd.

Furthermore, if $n$ is odd, then $G \in \mathbb{G}_1(n, r, 1)$ (r is odd) is the graph with the second-smallest $PI_w$ index $4n^2 - 2n - 2$. 


4 The upper bounds of $PI_w$ index of unicyclic graphs

In this section, the largest $PI_w$ index of unicyclic graphs and the extremal graphs will be obtained.

**Lemma 4.1.** Let $a_1, a_2, \ldots, a_r$ be $r$ positive integers and $\sum_{i=1}^{r} a_i = n$, then

$$a_1a_2 + a_2a_3 + \cdots + a_r a_1 \geq 2n - r.$$

The equality holds if and only if the ordered pairs $(a_1-1, a_2-1), (a_2-1, a_3-1), \ldots, (a_{r-1}-1, a_r-1), (a_r-1, a_1-1)$ are on $x$-axis or $y$-axis.

**Proof.** For each $i (1 \leq i \leq r)$, let $b_i = a_i - 1 \geq 0$, then $\sum_{i=1}^{r} b_i = n - r$, and

$$a_1a_2 + a_2a_3 + \cdots + a_r a_1 = (b_1 + 1)(b_2 + 1) + (b_2 + 1)(b_3 + 1) + \cdots + (b_r + 1)(b_1 + 1)$$
$$= (b_1 + b_2 + 1 + b_1b_2) + (b_2 + b_3 + 1 + b_2b_3) + \cdots + (b_r + b_1 + 1 + b_rb_1)$$
$$\geq 2(b_1 + b_2 + \cdots + b_r) + r$$
$$= 2n - r.$$

The equality holds if and only if $b_1b_2 + b_2b_3 + \cdots + b_{r-1}b_r + b_rb_1 = 0$, thus the results follows. \hfill \Box

**Theorem 4.1.** Let $n, r$ be integers with $3 \leq r \leq n$, $G \not\cong C_n$ be a connected unicyclic graph on $n$ vertices, and the length of unique cycle of $G$ be $r$, then

$$PI_w(G) \geq \begin{cases} n^3 - 2n^2r + 5n^2 + r^2n - rn, & \text{if } r \text{ is even}, \\ n^3 - 2n^2r + 5n^2 + r^2n - 6n - rn + 2r, & \text{if } r \text{ is odd}, \end{cases}$$

with the equality holds if and only if $G \in \mathcal{G}_2(n, r, 1)$.

**Proof.** There exists an integer $k$ ($1 \leq k \leq r \leq n$) such that $G = G(n, r, k)$ by $G \not\cong C_n$. Suppose $G_2(n, r, k) \in \mathcal{G}_2(n, r, k)$ is obtained from $G(n, r, k)$ by repeating $\beta$-transformation and $\gamma$-transformation. By Lemma 2.6, $PI_w(G(n, r, k)) < PI_w(G_2(n, r, k))$. Thus we only need to compute the value of $PI_w(G_2(n, r, k))$ for $k \geq 1$. For convenience, take $G_2 = G_2(n, r, k)$.

**Case 1:** $k = 1$.

**Subcase 1.1:** $r$ is even.

For any edge $e = uv \in E(G_2)$, we have $n_u(e) + n_v(e) = n$, and there are one vertex has degree $n - r + 2$, $n - r$ vertices have degree 1, $r - 1$ vertices have degree 2. By the
definition of \( PI_w \) index, we have

\[
PI_w(G_2) = n \cdot \left[ \sum_{uv \in E(G_1)} (\deg(u) + \deg(v)) \right] = n \cdot \left[ \sum_{u \in V(G_1)} \deg(u)^2 \right]
\]

\[
= n \cdot ((n - r + 2)^2 + 1^2(n - r) + 2^2(r - 1))
\]

\[
= n^3 - 2n^2r + 5n^2 + r^2n - nr
\]

**Subcase 1.2:** \( r \) is odd.

For any edge \( e = uv \in E(G_1 \setminus C_n) \), then \( n_u(e) + n_v(e) = n \). Note that \( m_1 = n - r + 1 \), by Proposition 3.2, there is only one edge \( e = uv \in E(C_r) \) satisfying \( n_u(e) + n_v(e) = n - (n - r + 1) = r - 1 \) whose weight is \( 2 + 2 = 4 \), and there are \( r - 1 \) edges \( e = xy \in E(C_r) \) satisfying \( n_x(e) + n_y(e) = n - 1 \), where 2 edges have weight \( 2 + (n - r + 2) = n - r + 4 \), and other \( r - 3 \) edges have weight \( 2 + 2 = 4 \). By the definition of \( PI_w \) index, we have

\[
PI_w(G_2) = 4(r - 1) + 4(r - 3)(n - 1) + 2(n - r + 4)(n - 1) + n \cdot (n - r + 3)(n - r)
\]

\[
= n^3 - 2n^2r + 5n^2 + r^2n - 6n - rn + 2r.
\]

**Case 2:** \( k \geq 2 \).

**Subcase 2.1** \( r \) is even.

For any edge \( e = uv \in E(G_2) \), then \( n_u(e) + n_v(e) = n \). Note that there are one vertex has degree \( m_i + 1 \) for each \( i \) \((1 \leq i \leq k)\), \( \sum_{i=1}^{k} (m_i - 1) \) vertices have degree 1, \( r - k \) vertices have degree 2, so by the definition of \( PI_w \) index, we have

\[
PI_w(G_2) = n \cdot \left[ \sum_{uv \in E(G_2)} (\deg(u) + \deg(v)) \right] = n \cdot \left[ \sum_{u \in V(G_2)} \deg(u)^2 \right]
\]

\[
= n \cdot \left[ \sum_{i=1}^{k} (m_i + 1)^2 + 1^2 \cdot \sum_{i=1}^{k} (m_i - 1) + 2^2(r - k) \right]
\]

\[
= n \cdot \left[ \sum_{i=1}^{k} (m_i - 1)^2 + 5 \cdot \sum_{i=1}^{k} (m_i - 1) + 4k + 4(r - k) \right]
\]

\[
\leq n \cdot \left[ \left( \sum_{i=1}^{k} (m_i - 1) \right)^2 + 5 \sum_{i=1}^{k} (m_i - 1) + 4k + 4(r - k) \right]
\]

\[
= n^3 - 2n^2r + 5n^2 + r^2n - rn.
\]

**Subcase 2.2:** \( r \) is odd.
For any edge \( e = uv \in E(G_2 \setminus C_n) \), then \( n_u(e) + n_v(e) = n \). By Proposition 3.2 and the definition of \( PI_w \) index, we have

\[
PI_w(G_2) = n \cdot \left( \sum_{i=1}^{k} (m_i + 2)(m_i - 1) \right) + \sum_{j=1}^{r} (l_j + l_{j+1} + 2) \left( n - l_j + \frac{r+1}{2} \right)
\]

\[
= n \cdot \left( \sum_{i=1}^{k} (m_i - 1)^2 \right) + 3n \cdot \left( \sum_{i=1}^{k} (m_i - 1) \right) + n \cdot \left( \sum_{j=1}^{r} (l_j + l_{j+1}) \right)
\]

\[
+ 2nr - \sum_{j=1}^{r} (l_j + l_{j+1} + 2)l_j + \frac{r+1}{4}
\]

\[
\leq n \cdot \left( \sum_{i=1}^{k} (m_i - 1)^2 \right) + 3(n - r)n + 2n^2 + 2nr - 2n
\]

\[
- \sum_{j=1}^{r} l_j l_{j+1} - \sum_{j=1}^{r} l_{j+1} l_j + \frac{r+1}{4}
\]

\[
\leq n(n - r)^2 + 3(n - r)n + 2n^2 + 2nr - 2n - (2n - r)(2n - r)
\]

\[
= n^3 - 2n^2r + 5n^2 + r^2n - rn - 6n + 2r.
\]

Since \( a_i \geq 1, i = 1, 2, \ldots, k \), then \( \sum_{i=1}^{k} a_i^2 \leq \left( \sum_{i=1}^{k} a_i \right)^2 = \sum_{i=1}^{k} a_i^2 + 2 \sum_{i \neq j} a_ia_j \) with the equality holds if and only if \( k = 1 \). We have \( m_i \geq 2, m_i - 1 \geq 1 \), then

\[
PI_w(G_2) < n^3 - 2n^2r + 5n^2 + r^2n - rn - 6n + 2r(k \geq 2).
\]

Combining the above two cases, we complete the proof.

Let \( n \) be a given positive integer, and \( f(r) = n^3 - 2n^2r + 5n^2 + r^2n - rn \), \( g(r) = n^3 - 2n^2r + 5n^2 + r^2n - 6n - rn + 2r \), then

\[
f'(r) = 2nr - 2n^2 - n, \quad g'(r) = 2nr - 2n^2 - n + 2.
\]

It is obvious that \( f'(r) < 0 \) and \( g'(r) < 0 \) when \( 3 \leq r \leq n \), so \( f(r) \) and \( g(r) \) are decreasing functions when \( 3 \leq r \leq n \). Thus we have

**Corollary 4.1.** Let \( n, r \) be integers with \( 3 \leq r \leq n \), \( G \not\sim C_n \) be a connected unicyclic graph on \( n \) vertices, the length of unique cycle of \( G \) be \( r \), then

\[
PI_w(G) \leq \begin{cases} 
PI_w(G_2(n, 3, 1)) = n^3 - n^2 + 6, & \text{if } r \text{ is odd}, \\
PI_w(G_2(n, 4, 1)) = n^3 - 3n^2 + 12n, & \text{if } r \text{ is even},
\end{cases}
\]

with the equality holds if and only if one of the following holds:
(1) if $r$ is odd, then $G \in \mathbb{G}_2(n, 3, 1)$ (see Fig. 13);
(2) if $r$ is even, then $G \in \mathbb{G}_2(n, 4, 1)$ (see Fig. 14).

By Proposition 3.1, Theorem 4.1 and Corollary 4.1, we have

**Theorem 4.2.** Let $n, r$ be integers and $n \geq 6$, $G$ be a connected unicyclic graph on $n$ vertices, and the length of the unique cycle of $G$ be $r$, then

$$PI_w(G) \leq n^3 - n^2 + 6,$$

with the equality holds if and only if $G \in \mathbb{G}_2(n, 3, 1)$.

**Remark.** Let $G$ be a connected unicyclic graph on $n$ vertices,

(1) if $n = 3$, then $G \cong C_3$ and $PI_w(G) = 24$;
(2) if $n = 4$, then $G \cong G_1(4, 3, 1)$ or $G \cong C_4$, and $PI_w(G_1(4, 3, 1)) = 54 < PI_w(C_4) = 64$;
(3) if $n = 5$, then $80 \leq PI_w(G) \leq 110$, and the first equality holds if and only if $G \cong C_5$, the second equality holds if and only if $G \cong G_2(5, 4, 1)$.

# 5 The vertex $PI$ index of unicyclic graphs

Let $G$ be an $n$-vertex graph and $n \geq 4$, then $PI(G) \leq |E(G)||V(G)|$, with equality holding if and only if $G$ is bipartite([14, 15]). The authors in [17] proved that $n(n - 1) \leq PI(G) \leq n \cdot \lceil \frac{n}{2} \rceil \cdot \lceil \frac{n}{2} \rceil$ and determined the extremal graphs. Recently in [8] the authors obtained the second-minimal and second-maximal graphs with respect to the vertex $PI$ index. In this section, we will show the vertex $PI$ index of unicyclic graphs is equal to $n^2$ or $n^2 - n$.

**Theorem 5.1.** Let $n, r$ be integers ($3 \leq r \leq n$) and $G$ be a connected unicyclic graph on $n$ vertices with the length of the unique cycle $r$, then
\[ PI(G) = \begin{cases} 
    n^2, & \text{if } r \text{ is even;} \\
    n^2 - n, & \text{if } r \text{ is odd.}
\end{cases} \]

**Proof.** If \( r \) is even, then \( G \) is bipartite, the result holds clearly.

If \( r \) is odd, then by Proposition 2.1 and Proposition 3.2, we have

\[
\begin{align*}
    PI(G) &= \sum_{e=uv \in E(G)} (n_u(e) + n_v(e)) \\
           &= \sum_{e=uv \in E(G \setminus C_r)} (n_u(e) + n_v(e)) + \sum_{e=uv \in E(C_r)} (n_u(e) + n_v(e)) \\
           &= \sum_{e=uv \in E(G \setminus C_r)} n + \sum_{i=1}^{r} (n - l_{i,r+1}) \\
           &= n(n - r) + nr - \sum_{i=1}^{r} l_{i,r+1} \\
           &= n^2 - n.
\end{align*}
\]

\[ \square \]

**Acknowledgments**

The authors would like to thank the referees for their useful comments and suggestions of this paper.

**References**


