Some New Sharp Bounds on the Distance Spectral Radius of Graph

Chang-Xiang He\textsuperscript{a}\textdagger, Ying Liu\textsuperscript{b}, Zhen-Hua Zhao\textsuperscript{c}

\textsuperscript{a}College of Science, University of Shanghai for Science and Technology, Shanghai, 200093, China
\textsuperscript{b}College of Mathematics and Information, Shanghai Lixin University of Commerce, Shanghai, 201620, China
\textsuperscript{c}School of Mathematics and Physics, Chongqing Institute of Technology, Chongqing, 400050, China

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Abstract: The $D$-eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of a graph $G$ are the eigenvalues of its distance matrix $D$ and form the $D$-spectrum of $G$ denoted by $\text{spec}D(G)$. The greatest $D$-eigenvalue is called the distance spectral radius of $G$, denoted by $\lambda_1$. In this paper we obtain some new lower and upper bounds for $\lambda_1$, and also show that all of our bounds are sharp.

1 Introduction

Let $G$ be a connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The distance matrix $D = D(G)$ of $G$ is defined so that its $(i, j)$-entry, $d_{ij}$, is equal to $d_G(v_i, v_j)$, the distance (length of the shortest path) between the vertices $v_i$ and $v_j$ of $G$. Then the distance matrix of a connected distance graph is irreducible and symmetric. The eigenvalues of $D(G)$ are said to be the $D$-eigenvalues of $G$ and form the $D$-spectrum of $G$, denoted by $\text{spec}D(G)$. Since the distance matrix is symmetric, all

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\textsuperscript{†}Corresponding author email: changxianghe@hotmail.com.
its eigenvalues $\lambda_i, i = 1, 2, \cdots, n$, are real and can be labeled so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The ordinary spectrum of $G$, which is the spectrum of the adjacency matrix of $G$ is well studied and many properties of graphs in connection with the spectrum are revealed during the past years. For details see the book [1] and the references cited therein. The greatest eigenvalue of the distance matrix of a graph $G$, $\lambda_1$ is called the distance spectral radius. For some recent works on distance spectrum of graphs, see [2–4]. In [3], the author gave some lower bounds for $\lambda_1$ and characterize those graphs for which these bounds are best possible. In this paper, we present some new lower and upper bounds for $\lambda_1$, and also prove that all of our bounds are sharp.

2 Main results and lemmas

**Definition 2.1** Let $G$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and a distance matrix $D = (d_{ij})$. Then the distance degree of $v_i$, denoted by $D_i$, is given by $D_i = \sum_{j=1}^{n} d_{ij}$.

**Definition 2.2** Let $G$ be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$, a distance matrix $D = (d_{ij})$, and a distance degree sequence $\{D_1, D_2, \cdots, D_n\}$. Then the second distance degree of $v_i$, denoted by $T_i$, is given by $T_i = \sum_{j=1}^{n} d_{ij}D_j$.

**Definition 2.3** Let $G$ be a graph with distance degree sequence $\{D_1, D_2, \cdots, D_n\}$ and second distance degree sequence $\{T_1, T_2, \cdots, T_n\}$. Then $G$ is pseudo $k$-distance regular if $\frac{T_i}{D_i} = k$ for all $1 \leq i \leq n$.

**Definition 2.4** Let $A$ be a matrix. We use $s_i(A)$ to denote the $i$th row sum of $A$.

The proof of Lemma 2.1 in [5] implies the following slightly stronger version.

**Lemma 2.1** [5] Let $A$ be a real symmetric $n \times n$ matrix, and let $\lambda$ be an eigenvalue of $A$ with an eigenvector $x$ all of whose entries are nonnegative. Then

$$\min_{1 \leq i \leq n} s_i(A) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(A).$$
Moveover, if all entries of \( x \) are positive then either of the equalities holds if and only if the row sums of \( A \) are all equal.

**Lemma 2.2** [6] Let \( A \) be a nonnegative irreducible \( n \times n \) matrix with spectral radius \( \lambda \). Then \( \lambda \) is a simple eigenvalue of \( A \), and if \( x \) is an eigenvector with eigenvalue \( \lambda \), then all entries of \( x \) are nonzero and have the same sign.

**Corollary 2.1** Let \( A \) be a nonnegative irreducible \( n \times n \) matrix with spectral radius \( \lambda \). Then

\[
\min_{1 \leq i \leq n} s_i(A) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(A).
\]

Equalities holds if and only if the row sums of \( A \) are all equal.

**Theorem 2.1** Let \( G \) be a connected graph with distance degree sequence \( \{D_1, D_2, \ldots, D_n\} \), second distance degree sequence \( \{T_1, T_2, \ldots, T_n\} \), and distance spectral radius \( \lambda_1 \). Then

\[
\min \{m_i : 1 \leq i \leq n\} \leq \lambda_1 \leq \max \{m_i : 1 \leq i \leq n\}.
\]

(1)

where \( m_i = \frac{T_i}{D_i} \). Moveover, any equality holds if and only if \( G \) is pseudo distance regular.

**Proof.** Let \( M = \text{diag}(D_1, \ldots, D_n) \). Then \((i,j)\)-entry of \( M^{-1}DM \) is \( \frac{d_i D_j}{D_i} \), and

\[
s_i(M^{-1}DM) = \frac{T_i}{D_i} = m_i \quad (1 \leq i \leq n).
\]

It is not difficult to see that \( M^{-1}DM \) is a nonnegative irreducible \( n \times n \) matrix with spectral radius \( \lambda_1 \). Now we use Corollary 2.1 by taking \( A = M^{-1}DM \), the desired result holds.

Now we assume that \( G \) is pseudo distance regular, then \( m_i = \frac{T_i}{D_i} = k \) for all \( i \), and hence \( \min \{m_i : 1 \leq i \leq n\} = \max \{m_i : 1 \leq i \leq n\} = k \). Thus both of the equalities hold.

Conversely, if one of the equalities holds, by Corollary 2.1, the row sums of \( M^{-1}DM \) are all equal. That is, \( m_i = \frac{T_i}{D_i} \) \((1 \leq i \leq n)\) are all equal, which may implies that \( G \) is a pseudo distance regular graph. \( \Box \)
Theorem 2.2 Let $G$ be a connected graph with second distance degree sequence $\{T_1, T_2, \cdots, T_n\}$, and distance spectral radius $\lambda_1$. Then

$$\min\{\sqrt{T_i} : 1 \leq i \leq n\} \leq \lambda_1 \leq \max\{\sqrt{T_i} : 1 \leq i \leq n\}.$$  \hspace{1cm} (2)

Moreover, any equality holds if and only if $G$ has same value of $T_i$ for all $i$.

Proof. Let $D = (d_{ij})$ be the distance matrix of $G$ and $\{D_1, D_2, \cdots, D_n\}$ be the distance degree sequence of $G$. Since $(D^2)_{ij} = \sum_{k=1}^{n} d_{ik}d_{kj}$, we have

$$s_i(D^2) = \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ik}d_{kj}$$

$$= \sum_{k=1}^{n} d_{ik} \sum_{j=1}^{n} d_{kj}$$

$$= \sum_{k=1}^{n} d_{ik}D_k$$

$$= T_i$$

Let $x$ be an eigenvector corresponding to $\lambda_1$, all of whose entries are positive, that is, $Dx = \lambda_1 x$, then $D^2x = \lambda_1^2 x$. By Lemma 2.1,

$$\min\{T_i : 1 \leq i \leq n\} \leq \lambda_1^2 = \lambda(D^2) \leq \max\{T_i : 1 \leq i \leq n\}.$$  

Thus $\min\{\sqrt{T_i} : 1 \leq i \leq n\} \leq \lambda_1 \leq \max\{\sqrt{T_i} : 1 \leq i \leq n\}$.

Now we assume that $G$ has same value of $T_i$ for all $i$, then $\min\{\sqrt{T_i} : 1 \leq i \leq n\} = \max\{\sqrt{T_i} : 1 \leq i \leq n\}$, both of the equalities hold.

Conversely, if one of the equalities holds, that is, $\lambda_1^2 = \min\{T_i : 1 \leq i \leq n\}$ or $\lambda_1^2 = \max\{T_i : 1 \leq i \leq n\}$. By Corollary 2.1, $s_i(D^2) = T_i^2$ (1 $\leq i \leq n$) all are equal. So $G$ has same value of $T_i$ for all $i$.

\[\Box\]

Theorem 2.3 Let $G$ be a connected graph of order $n$, and $\lambda_1$ be the distance spectral radius, then

$$\lambda_1 \leq \max\{\sqrt{m_im_j} : 1 \leq i, j \leq n\},$$  \hspace{1cm} (3)
where \( m_i = \frac{T_i}{D_i} \). Moreover, the equality holds if and only if \( G \) is a pseudo distance regular graph.

**Proof.** Let \( M = \text{diag}(D_1, \ldots, D_n) \) and \( x = (x_1, x_2, \ldots, x_n)^T \) be an eigenvector of \( M^{-1}DM \) corresponding to the eigenvalue \( \lambda_1 \). Also let one entry, say \( x_i \), be equal to 1 and the other entries be less than or equal to 1, that is, \( x_i = 1 \) and \( 0 \leq x_k \leq 1 \) for any \( k \). Let \( x_j = \max\{x_k : k \neq i\} \).

Now the \((i, j)\)-entry of \( M^{-1}DM \) is \( \frac{d_{ij}D_j}{D_i} \), and

\[
M^{-1}DMx = \lambda_1 x
\]

From the \( i \)th equation of (4),

\[
\lambda_1 x_i = \sum_{k=1}^{n} \frac{d_{ik}D_kx_k}{D_i} \\
= \frac{1}{D_i} \sum_{k=1}^{n} d_{ik}D_kx_k \\
\leq \frac{d_{ii}D_ix_i}{D_i} + \frac{x_j}{D_i} \sum_{k=1, k \neq i}^{n} d_{ik}D_k \\
= \frac{x_j}{D_i} \sum_{k=1, k \neq i}^{n} d_{ik}D_k \\
= \frac{x_j}{D_i} \sum_{k=1}^{n} d_{ik}D_k \\
= \frac{T_i}{D_i} x_j = m_i x_j \quad (5)
\]

From the \( j \)th equation of (4),

\[
\lambda_1 x_j = \sum_{k=1}^{n} \frac{d_{jk}D_kx_k}{D_j} \\
= \frac{1}{D_j} \sum_{k=1}^{n} d_{jk}D_kx_k \\
\leq \frac{1}{D_j} \sum_{k=1}^{n} d_{jk}D_k \\
= \frac{T_j}{D_j} \\
= m_j \quad (6)
\]
Combing (5), (6) and $x_i = 1$, we get $\lambda_i^2 \leq m_i m_j$.

Therefore, $\lambda_1 \leq \sqrt{m_i m_j} \leq \max\{\sqrt{m_i m_j} : 1 \leq i, j \leq n\}$.

Now we assume that $G$ is pseudo distance regular, so $\frac{T_i}{D_i} = k$ or $T_i = k D_i$ for all $i$. Then

$$D(D_1, D_2, \cdots, D_n)^T = k(D_1, D_2, \cdots, D_n)^T$$

showing that $(D_1, D_2, \cdots, D_n)^T$ is an eigenvector corresponding to $k$. Note that

$\lambda_1 \leq \sqrt{k^2}$, we have $\lambda_1 = k$. Thus the equality holds.

Conversely, if $\lambda_1$ attains the upper bound then all equalities in the above argument must hold. In particular, from (6) that $x_k = 1$, for $1 \leq k \leq n$, that is, $x = (1, 1, \cdots, 1)^T$. Hence $M^{-1} D M (1, 1, \cdots, 1)^T = \lambda_1 (1, 1, \cdots, 1)^T$, this then implies that

$$\frac{T_k}{D_k} = \frac{\sum_{j=1}^n d_{kj} D_j}{D_j} = \lambda_1$$

or in other words $G$ is pseudo distance regular. \hfill \qed

References


