MODIFIED NARUMI–KATAYAMA INDEX

Modjtaba Ghorbani,¹ Mahin Songhori,¹
Ivan Gutman²

¹Department of Mathematics, Faculty of Science,
Shahid Rajaee Teacher Training University,
Tehran, 16785–136, Iran
e-mail: modjtaba.ghorbani@gmail.com

²Faculty of Science, University of Kragujevac,
P. O. Box 60, 34000 Kragujevac, Serbia
e-mail: gutman@kg.ac.rs

(Received January 5, 2012)

ABSTRACT. The Narumi–Katayama index of a graph $G$ is equal to the product of the
degrees of the vertices of $G$. In this paper we consider a new version of the Narumi–Katayama index in which each vertex degree $d$ is multiplied $d$ times. We characterize the graphs extremal w.r.t. this new topological index.

1 Introduction

A topological index is a graph invariant used in structure–property correlations. Hundreds of topological indices have been introduced and studied [1], starting with the seminal work by Wiener in which he used the sum of all shortest–path distances of a (molecular) graph for modeling physical properties of alkanes [2]. The aim of this paper is to put forward a new variant of the Narumi–Katayama index. We determine its basic properties and characterize graphs extremal with respect to it.
2 Definitions and preliminaries

Our notation is standard and mainly taken from standard books of graph theory such as, e. g., [3]. All graphs considered in this paper are simple and connected. The vertex and edge sets of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices of $G$ is denoted by $n$.

The degree $d_v$ of a vertex $v \in V(G)$ is the number of vertices of $G$ adjacent to $v$. A vertex $v \in V(G)$ is said to be isolated, pendent, or fully connected if $d_v = 0$, $d_v = 1$, or $d_v = n - 1$, respectively. The $n$-vertex graph in which all vertices are fully connected is the complete graph $K_n$. The $n$-vertex graph with a single fully connected vertex and $n - 1$ pendent vertices is the star $S_n$. The connected $n$-vertex graph with two pendent vertices and $n - 2$ vertices of degree 2 is the path $P_n$. The connected $n$-vertex graph whose all vertices are of degree 2 is the cycle $C_n$ [3].

In the 1980s, Narumi and Katayama considered the product

$$NK = NK(G) = \prod_{u \in V(G)} d_u$$

and named it “simple topological index” [4]. Eventually this molecular structure-descriptor was re-named as “Narumi–Katayama index” [5]. The properties of $NK$ were much investigated, see [4–11].

We now propose a modified version of the Narumi–Katayama index as follows:

$$NK^* = NK^*(G) = \prod_{u \in V(G)} d_u^{d_u}.$$ 

3 Main Results

In this section we present the value of $NK^*$ for several classes of graphs.

Example 1. Let $S_n$ be the star graph on $n$ vertices. Its central vertex has degree $n - 1$ and its other vertices are pendent. This implies

$$NK^*(S_n) = (n - 1)^{n-1}.$$ 

Example 2. Let $K_n$ be the complete graph on $n$ vertices. All vertices of $K_n$ have degree $n - 1$ and so $NK^*(K_n) = (n - 1)^{n(n-1)}$. 
Example 3. Let $P_n$ be the path with $n$ vertices. The pendent vertices have degree 1 and other vertices have degree two. Hence,

$$NK^*(P_n) = 2^{2(n-2)} = 4^{n-2}.$$ 

Example 4. Consider the cycle $C_n$ with $n$ vertices. Since its every vertex is of degree 2, then

$$NK^*(C_n) = 2^{2n} = 4^n.$$ 

Theorem 5. Let $G$ be an arbitrary $n$-vertex graph. Then

$$NK^*(G) \leq NK^*(K_n)$$

with equality if and only if $G \cong K_n$.

Theorem 6.

$$NK^*(G) = \prod_{uv \in E(G)} d_u d_v.$$ 

Proof. For every vertex $u \in V(G)$, $d_u$ appears $d_u$ times in the product $\prod_{uv \in E(G)} d_u d_v$. 

Theorem 7.

(a) The $n$-vertex tree with maximal modified Narumi–Katayama index is the star $S_n$. Thus, $NK^*(T) < (n-1)^{n-1}$ for any $n$-vertex tree $T$ different from $S_n$.

(b) The $n$-vertex connected unicyclic graph with maximal Narumi–Katayama index is $S_n + e$, depicted in Fig. 1; $NK^*(S_n + e) = 16 (n-1)^{n-1}$. Thus, $NK^*(U) < 16 (n-1)^{n-1}$ for any $n$-vertex connected unicyclic graph $U$ different from $S_n + e$.

(c) Among all connected bicyclic graphs on $n$ vertices, the graph $S_n + e + e'$, depicted in Fig. 2, has the maximal modified Narumi–Katayama index; $NK^*(S_n + e + e') = 256 (n-1)^{n-1}$. Thus, $NK^*(B) < 256 (n-1)^{n-1}$ for any $n$-vertex connected bicyclic graph $B$ different from $S_n + e + e'$. 

Fig. 1. The unicyclic graph $S_{n} + e$ with maximal $NK^*$-value.

Fig. 2. The bicyclic graph $S_{n} + e + e'$ with maximal $NK^*$-value.

Theorem 8.

(a) The $n$-vertex tree with minimal modified Narumi–Katayama index is the path $P_{n}$. Thus, $NK^*(T) > 4^{n-2}$ for any $n$-vertex tree $T$ different from $P_{n}$.

(b) The $n$-vertex connected unicyclic graph with minimal Narumi–Katayama index is the cycle $C_{n}$. Thus, $NK^*(U) > 4^{n}$ for any connected $n$-vertex unicyclic graph $U$ different from $C_{n}$.

(c) Among all connected bicyclic graphs on $n$ vertices, the graphs $B_{min}$ whose structure is indicated in Fig. 3, have minimal modified Narumi–Katayama index; $NK^*(B_{min}) = 2^{2n+6}$. Thus, $NK^*(B) > 2^{2n+6}$ for any connected $n$-vertex bicyclic graph $B$ whose structure is different from $B_{min}$. 
Fig. 3. The bicyclic graph $B_{\min}$ with minimal $NK^*$-value. Recall that there exist $\lfloor (n - 3)/2 \rfloor$ distinct $n$-vertex bicyclic graphs of the type $B_{\min}$.

**Theorem 9.**

(a) The $n$-vertex tree with second–maximal modified Narumi–Katayama index is the graph $S'_n$, depicted in Fig. 4; $NK^*(S'_n) = 4(n - 2)^{n-2}$. Thus, $NK^*(T) < 4(n - 2)^{n-2}$ for any $n$-vertex tree $T$ different from $S_n$ and $S'_n$.

(b) The $n$-vertex connected unicyclic graph with second–maximal modified Narumi–Katayama index is the graph $K_n$, depicted in Fig. 5; $NK^*(K_n) = 64(n - 2)^{n-2}$. Thus, $NK^*(U) < 64(n - 2)^{n-2}$ for any connected $n$-vertex unicyclic graph $U$ different from $S_n + e$ and $K_n$.

(c) Among all connected bicyclic graphs on $n$ vertices, the graph $F_n$, depicted in Fig. 6, has the second–maximal modified Narumi–Katayama index; $NK^*(F_n) = 2^{10}(n - 2)^{n-2}$. Thus, $NK^*(B) < 2^{10}(n - 2)^{n-2}$ for any connected $n$-vertex bicyclic graph $B$ different from $S_n + e + e'$ and $K_n$.

Fig. 4. The tree $S'_n$ with second–maximal $NK^*$-value.
Fig. 5. The unicyclic graph \( K_n \) with second-maximal \( NK^* \)-value.

Fig. 6. The bicyclic graph \( F_n \) with second-maximal \( NK^* \)-value.

**Theorem 10.**

(a) The \( n \)-vertex tree with second-minimal modified Narumi–Katayama index is one of the trees \( T_{a,b,c} \), depicted in Fig. 7, where \( a, b, c \geq 1 \) and \( a + b + c = n - 1 \); \( NK^*(T_{a,b,c}) = 27 \cdot 4^{n-4} \). Thus, \( NK^*(T) > 27 \cdot 4^{n-4} \) for any \( n \)-vertex tree \( T \) different from \( P_n \) and \( T_{a,b,c} \).

(b) The \( n \)-vertex connected unicyclic graph with second-minimal modified Narumi–Katayama index is the graph \( R_n \), depicted in Fig. 8; \( NK^*(R_n) = 27 \cdot 4^{n-2} \). Thus, \( NK^*(T) > 27 \cdot 4^{n-2} \) for any \( n \)-vertex connected unicyclic graph \( U \) different from \( C_n \) and \( R_n \).

(c) Among all connected bicyclic graphs on \( n \) vertices, the graphs \( B'_{\text{min}} \), whose structure is indicated in Fig. 9 have second-minimal modified Narumi–Katayama index; \( NK^*(B'_{\text{min}}) = 36 \cdot 4^{n-2} \). Thus, \( NK^*(B) > 36 \cdot 4^{n-2} \) for any connected \( n \)-vertex bicyclic graph \( B \) whose structure is different from \( B_{\text{min}} \) and \( B'_{\text{min}} \).
Fig. 7. The tree $T_{a,b,c}$ with second–minimal $NK^*$-value.

$$a+b+c+1=n$$

Fig. 8. The unicyclic graph $R_n$ with second–minimal $NK^*$-value.

Fig. 9. The bicyclic graph $B'_{min}$ with second–minimal $NK^*$-value. Recall that there exist $\lfloor (n-4)/2 \rfloor$ distinct $n$-vertex bicyclic graphs of the type $B'_{min}$.

REFERENCES


